Calculus | Derivatives, Integrals, and Identities

Derivative Rules

Let $f(x)$ and $g(x)$ be continuous functions. Let c be some constant.

Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$, for $n \in \mathbb{R}$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Sum and Difference: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Constant Multiple: $\frac{d}{dx}[cf(x)] = cf'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

Exponential/Logarithmic Functions

Derivatives

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}a^x = a^x \cdot \ln(a)$, for $a > 0$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$, for $x > 0$

$\frac{d}{dx}\ln|x| = \frac{1}{x}$, for $x \neq 0$

$\frac{d}{dx}\log_a x = \frac{1}{x\ln(a)}$

Properties

Let A and B be positive real numbers. Let c be a constant.

$\log_b A + \log_b B = \log_b(AB)$

$\log_b A - \log_b B = \log_b\left(\frac{A}{B}\right)$

$c\log_b A = \log_b(A^c)$

Trigonometric Functions

Trigonometric

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\cot(x) = -\csc^2(x)$

$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$

Hyperbolic

$\frac{d}{dx}\sinh(x) = \cosh(x)$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\frac{d}{dx}\tanh(x) = \text{sech}^2(x)$

$\frac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x)$

$\frac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x)$

$\frac{d}{dx}\coth(x) = -\text{csch}^2(x)$

$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$

$\frac{d}{dx}\text{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{1+x^2}}$

$\frac{d}{dx}\text{sech}^{-1}(x) = \frac{-1}{|x|\sqrt{1-x^2}}$

$\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$

Integrals

Fundamental Theorem of Calculus

Part 1:
If $f(x)$ is continuous on $[a, b]$ then $g(x) = \int_a^x f(t)dt$ is also continuous on $[a, b]$ and
$g'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Part 2:
$f(x)$ is continuous on $[a, b]$, $F(x)$ is an anti-derivative of $f(x)$ where $F(x) = \int f(x)dx$ and
$\int_a^b f(x)dx = F(b) - F(a)$

Exponential Integrals:

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C, (n \neq -1)$$

$$\int k \, dx = kx + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \ln(x) = x\ln(x) - x + C$$

Average Function Value:
The average value of $f(x)$ on $a \leq x \leq b$ is:
$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

Work:
If a force of $F(x)$ moves an object in $a \leq x \leq b$, the work done is: $W = \int_a^b F(x)dx$

Properties:

Additive: $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Constant Multiple: $\int kf(x)dx = k\int f(x)dx$

Substitution: $\int f(g(x))g'(x)dx = \int f(u)du$
(where $u = g(x)$)

Integration by Parts: $\int udv = uv - \int vdu$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any value c

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$$

Arc Length (L) and Surface Area:

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$, if $y = f(x), a \leq x \leq b$

$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$, if $x = h(y), c \leq y \leq d$

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$, if
$x = f(t), y = g(t), a \leq t \leq b$

Surface Area rotated about the x-axis: $\int_a^b 2\pi y ds$

Surface Area rotated about the y-axis: $\int_a^b 2\pi x ds$

Approximating Definite Integrals:

Let f be a continuous function on the interval $[a, b]$. Given an integral $\int_a^b f(x)dx$ and some n, divide $[a, b]$ into n equal sub-intervals of width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a$, $x_n = b$, and $x_i = a + \Delta x \cdot i$.

<u>Left Endpoint Definition:</u> $\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

<u>Right Endpoint Definition:</u> $\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i)\Delta x$

<u>Midpoint Rule:</u> $\int_a^b f(x)dx \approx \Delta x\,[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)]$, where $f(x_i^*)$ is midpoint $[x_{i-1}, x_i]$

<u>Trapezoid Rule:</u> $\int_a^b f(x)dx \approx \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]$

<u>Simpson's Rule:</u> (For n even) $\int_a^b f(x)dx \approx \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$

Trigonometry:

$$\int \cos(x)\ dx = \sin(x) + C$$

$$\int \sin(x)\ dx = -\cos(x) + C$$

$$\int \tan(x)\ dx = \ln|\sec(x)| + C$$

$$\int \sec^2(x)\ dx = \tan(x) + C$$

$$\int \csc^2(x)\ dx = -\cot(x) + C$$

$$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$$

$$\int \sec(x)\tan(x)\ dx = \sec(x) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\ dx = \tan^{-1}\left(\frac{x}{a}\right) + C$$

Hyperbolic Trigonometry:

$$\int \cosh(x)\ dx = \sinh(x) + C$$

$$\int \sinh(x)\ dx = \cosh(x) + C$$

$$\int \sinh^2(x)\ dx = \tanh(x) + C$$

$$\int \operatorname{csch}^2(x)\ dx = -\coth(x) + C$$

$$\int \operatorname{sech}(x)\tanh(x)\ dx = -\operatorname{sech}(x) + C$$

$$\int \operatorname{csch}(x)\coth(x)\ dx = -\operatorname{csch}(x) + C$$

Trig Substitutions:

Given:	$\sqrt{a^2 - b^2x^2}$	$\sqrt{b^2x^2 - a^2}$	$\sqrt{a^2 + b^2x^2}$
Transformation:	$x = \frac{a}{b}\sin\theta$	$x = \frac{a}{b}\sec(\theta)$	$x = \frac{a}{b}\tan(\theta)$
Result:	$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2(\theta) = \sec^2(\theta) - 1$	$\sec^2(\theta) = 1 + \tan^2(\theta)$

Trigonometric Identities

Basic and Pythagorean:

$$\sin(\theta) = \frac{1}{\csc(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

Angle Sum and Difference:

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

Law of Sines and Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Double Angle:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 1 - 2\sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\end{aligned}$$

Half Angle:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$$

Power Reduction:

$$\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$$

$$\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum to Product:

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos(\alpha) \pm \cos(\beta) = \pm 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\tan(\alpha) \pm \tan(\beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha)\cos(\beta)}$$