Calculus Introduction | Calculus I • Math 1210

Limits

Description

Limits describe how a function behaves near a point, instead of at that point, approached from the left ($x \to a^-$), right ($x \to a^+$), or both directions ($x \to a$).

Continuity:
A function f is continuous at a point a if and only if:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a} f(x)$ exists if: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Theorems

Additive: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

Product: $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

Quotient: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

Constant Multiple: $\lim_{x \to a} c[f(x)] = c \cdot \lim_{x \to a} f(x)$

Exponent: $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

Constant: $\lim_{x \to a} c = c$

Limit of Continuous Functions: For $g(x)$ continuous, $\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x))$

Derivatives

Definition

Derivatives of a function describe the function's instantaneous rate of change and the slope of the tangent line to the function's graph at a particular point.

Limit Definition of a Derivative:

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative at a point:

$$\frac{d}{dx} f(x)\bigg|_{x=a} = f'(x)\bigg|_{x=a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Theorems

Constant: $\left(af(x)\right)' = af'(x)$

Additive: $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$

Product: $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

Quotient: $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Chain: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

L'Hopital's Theorem

Indeterminate Forms

L'Hôpital's rule helps us evaluate indeterminate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Other examples of indeterminate forms are when the limit evaluates to $0 \cdot \infty$, 0^0, or 1^∞. If L'Hôpital's rule is applied and the result is still indeterminate, repeat the process of L'Hôpital's rule until a definite limit is found.

L'Hopital's Theorem

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $g(x) \neq 0$, Or, if $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} f(x) = \pm\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Optimization

Overview

Use derivative tests to determine local extrema. Test at a critical point, c, where $f'(x) = 0$ or is undefined.

First Derivative Test

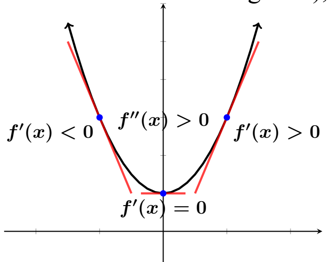
Examine the derivative behavior just around c.

- Negative to positive implies a local minimum
- Positive to negative implies a local maximum
- If both are the same (both positive or both negative), then there is no extremum

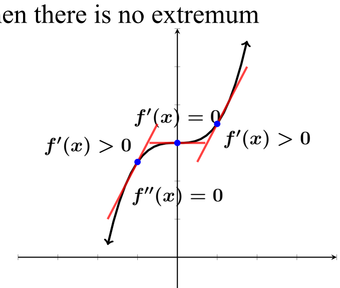
Second Derivative Test

Examine the second derivative at c.

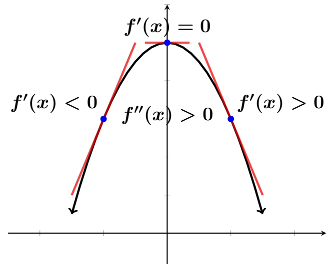
- $f''(c) > 0$ implies a local minimum
- $f''(c) < 0$ implies a local maximum
- $f''(c) = 0$ is inconclusive (use results from first derivative test instead)



Local Minimum



Neither Minimum nor Maximum



Local Maximum

Integrals

Definition	Theorems
Given $F'(x) = f(x)$, Evaluate definite integrals using The Fundamental Theorem of Calculus, $\int_b^a f(x) = F(b) - F(a)$ Evaluate indefinite integrals as: $\int f(x) = F(x) + C$, where C is some constant.	Additive: $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$ Constant Multiple: $\int kf(x)dx = k\int f(x)dx$ Substitution: $\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$ Integration by Parts: $\int udv = uv - \int vdu$

Relationships and Theorems

Concept	Description
Area	If we approximate the area under $f(x)$ between $x = a$ and $x = b$ by dividing into n rectangles, then the area is approximately equal to $\sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. As $n \to \infty$, this approaches $\int_a^b f(x)dx$.
Mean Value Theorem	Where $f(x)$ is continuous and differentiable on an interval $[a, b]$ there exists some c within $[a, b]$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$. There exists some point where the instantaneous slope is equal to the average slope from a to b.
Extreme Value Theorem	If f is a continuous function in a closed interval $[a, b]$ then f achieves both an absolute maximum and minimum in $[a, b]$. Furthermore, the absolute extreme occur at a or b or at a critical number between a and b
Average Function Value	On interval $[a, b]$, the average value for $f(x)$ is $\frac{1}{b-a}\int_a^b f(x)dx$.
Arc Length	The length of a curve on some interval $[a, b]$ is $\int_a^b \sqrt{1 + (f'(x))^2}dx$.
Position, Velocity, and Acceleration	With respect to time, the position, velocity, and acceleration functions can be related to each other using derivatives. Position: $s(t)$, Velocity: $v(t) = \frac{d}{dt}(s(t))$, Acceleration: $a(t) = \frac{d}{dt}(v(t)) = \frac{d^2}{dt^2}(s(t))$

Common Derivatives

$\frac{d}{dx}a = 0$

$\frac{d}{dx}x^n = nx^{n-1}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}a^{g(x)} = \ln(a)a^{g(x)}g'(x)$

$\frac{d}{dx}\log_b(x) = \frac{1}{\ln(b)} \cdot \frac{1}{x}$

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\cot(x) = \csc^2(x)$

$\frac{d}{dx}\csc(x) = \csc(x)\cot(x)$

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\text{arccot}(x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\text{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Common Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, (n \neq -1)$

$\int k\,dx = kx + C$

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int \cos(x)\,dx = \sin(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$

$\int \tan(x)\,dx = \ln|\sec(x)| + C$

$\int \sec^2(x)\,dx = \tan(x) + C$

$\int \sec(x)\tan(x)\,dx = \sec(x) + C$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

$\int \frac{1}{a^2+x^2}\,dx = \arctan\left(\frac{x}{a}\right) + C$