UtahStateUniversity
AggieMathLearningCenter

Series Test for Convergence, Divergence | Calculus II • Math 1220

Divergence Test

How to Use	Conclusions	Notes
Use divergence test if $\lim_{n\to\infty} a_n \neq 0$	If $\lim_{n\to\infty} a_n \neq 0$ or DNE series diverges. If $\lim_{n\to\infty} a_n = 0$ test inconclusive.	• This test does not show convergence and can be used with alternating series

Geometric Series Test

How to Use	Conclusions	Notes
Geometric series have form $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$ Diverges if $\lvert r\rvert \geq 1$	• Useful if n is only in the exponent. Simplification may be needed • This is the **ONLY** test that tells us what a series converges to.

P-Series Test

When to Use	Conclusions	Notes
P-series are in the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$ the series converges If $p \leq 1$ the series diverges	• Useful for comparison tests

Integral Test

When to Use	Conclusions	Notes
Suppose $f(x)$ is: ☐ continuous on $[k, \infty)$ ☐ positive on $[k, \infty)$ ☐ Ultimately decreasing ☐ $a_n = f(n)$	If $\int_k^{\infty} a_n$ converges, series converges. If $\int_k^{\infty} a_n$ diverges, series diverges.	• This test is never inconclusive. • The value of the integral is NOT the value of the series!

Alternating Series Test

When to Use	Conclusions	Notes
☐ In form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ ☐ a_n is always positive ☐ Ultimately decreasing, $a_n \geq a_{n+1}$ for all $n > N$ ☐ $\lim_{n\to\infty} a_n = 0$	Test inconclusive without all conditions met. BUT if all met, series converges.	• $\sum a_n$ converges absolutely if $\sum \lvert a_n\rvert$ converges. • If $\sum a_n$ converges and $\sum \lvert a_n\rvert$ diverges, $\sum a_n$ converges conditionally.

Comparison Test

When to Use	Conclusions	Notes
Let $\sum a_n$ and $\sum b_n$ be series with positive terms. Use when a series has a similar form to a p-series or a geometric series	If $\sum a_n \leq \sum b_n$ for all n sufficiently large and $\sum b_n$ converges, then $\sum a_n$ converges. If $\sum a_n \geq \sum b_n$ for all n sufficiently large and $\sum b_n$ diverges, then $\sum a_n$ diverges	• Use to compare a complicated series with a less-complicated series that grows at the same rate. • Need to come up with b_n

Limit Comparison Test

When to Use	Conditions	Conclusions
Use to show that the series $\sum a_n$ and $\sum b_n$ are growing at the same rate.	Let $\sum a_n$ and $\sum b_n$ be series with positive terms. $\sum b_n$ is a known series.	• If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$, both diverge or converge. • If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, $\sum a_n$ converges. • If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, $\sum a_n$ diverges.

Ratio Test

When to Use	How to Use	Conclusions
Use when we have n's in the exponents and connected to constants. This is also useful for factorials ($n!$)	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = L$	• If $L < 1$, then $\sum a_n$ converges absolutely. • If $L > 1$, then $\sum a_n$ diverges. • If $L = 1$, then the test is inconclusive.

Root Test

When to Use	How to Use	Conclusions
Useful if the series has a power of n. This test is not commonly used.	$\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} = L$	• If $L < 1$, then $\sum a_n$ converges absolutely. • If $L > 1$, then $\sum a_n$ diverges. • If $L = 1$, then the test is inconclusive.

Remainder Estimate Inequalities

Alternating Series Remainder

$$R_n = \left\lvert \sum_{n=1}^{\infty}(-1)^{n+1}a_n - \sum_{n=1}^{N}(-1)^{n+1}a_n \right\rvert \leq a_{N+1}$$

(from Alternating Series Remainder)

$$\lvert R_n\rvert \leq a_{n+1}$$

Integral Remainder

$$\int_{n+1}^{\infty} f(x)dx < R_n < \int_{n}^{\infty} f(x)dx$$

Taylor's Theorem with Remainder

$$\lvert R_n\rvert \leq \frac{M}{(n+1)!}\lvert x-c\rvert^{n+1}$$

Taylor Series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Common Maclaurin Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$$