Vector Calculus | Multivariate Calculus • Math 2210

Definitions

Gradient	Divergence	Curl	Jacobian
$F:\mathbb{R}^n\to\mathbb{R}$ $\nabla f(x,y,z)$ $=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$	$F=\langle P,Q,R\rangle$ $F:\mathbb{R}^3\to\mathbb{R}^3$ div $\mathbf{F}=P_x+Q_y+R_z$ $=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$	$F=\langle P,Q,R\rangle$ $F:\mathbb{R}^3\to\mathbb{R}^3$ curl $\mathbf{F}=(R_y-Q_z)\mathbf{i}+$ $(P_z-R_x)\mathbf{j}+(Q_x-P_y)\mathbf{k}$	$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$ $\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$

Work, Flux and Circulation

Work	Flux	Circulation
$\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_a^b \mathbf{F}(r(t))\cdot r'(t)\,dt$	$\int_C \mathbf{F}\cdot\mathbf{N}\,ds=\int_a^b \mathbf{F}(r(t))\cdot n(t)\,dt$ where $\mathbf{N}=\frac{n(t)}{\lVert n(t)\rVert}$, $n(t)=\langle y'(t),-x'(t)\rangle$	$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ $\mathbf{T}$ is a unit vector wrt arc length

Fundamental Theorem of Line Integrals

Description	Theorem
Simplifies line integrals to evaluating the potential function f at the endpoints and subtracting. The curve is parameterized by $\mathbf{r}(t)$ where $\mathbf{r}(a)$ is initial and $\mathbf{r}(b)$ is final.	$\int_C \nabla f\cdot d\mathbf{r}=f(\mathbf{r}(b))-f(\mathbf{r}(a))$

Potential Functions

$\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$
A vector field is conservative if it has a potential function.

How to find one:
1: $\int P(x,y)dx=g(x,y)+h(y)$
2: $\frac{\partial}{\partial y}(g(x,y)+h(y))=g_y(x,y)+h'(y)$
3: $g_x(x,y)+h'(y)=Q(x,y)$
4: $\int h'(y)dy=h(y)$
5: $f(x,y)=g(x,y)+h(y)+c$

Green's and Stoke's Theorems

Description	Theorems
Green's Thm. converts the circulation of a vector field $F=\langle P,Q\rangle$ along a simple closed curve C to a double integral over region D. Stoke's Thm. is a generalized Green's Thm. not bounded to a plane. $\mathbf{r}(t)$ parametrizes C (boundary of S) and $\mathbf{N}$ is a unit normal vector to the surface.	Green's Circulation: $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_D (Q_x-P_y)\,dA=\iint_D$ curl $\mathbf{F}\,dA$ Green's Flux: $\oint_C \mathbf{F}\cdot\mathbf{N}ds=\iint_D (P_x+Q_y)dA=\iint_D$ div $\mathbf{F}\,dA$ Stoke's Theorem: $\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S$ curl $\mathbf{F}\cdot d\mathbf{S}$

Divergence Theorem

Description	Theorem
Transforms difficult surface integral over S into a triple integral over solid E using the divergence of vector field F. Bounds are usually found with change of coordinates.	$\iiint_E$ div $\mathbf{F}\,dV=\iint_S \mathbf{F}\cdot d\mathbf{S}$

Applications

Optimization

Second Derivative Test

$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2$
- If $D>0$ and $fxx(x_0,y_0)>0$, (x_0,y_0) is a local min
- If $D>0$ and $fxx(x_0,y_0)<0$, (x_0,y_0) is a local max
- If $D<0$, (x_0,y_0) is a saddle point
- If $D=0$, test is inconclusive

Lagrange Multipliers

Optimize $f(x,y,z)$ subject to constraint $g(x,y,z)=k$
$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ so $f_x=\lambda g_x$, $f_y=\lambda g_y$, $f_z=\lambda g_z$
Solve system of equations, substitute solutions into $f(x,y,z)$, and identify minimum, maximum (need existence and $\nabla g(x,y,z)\neq 0$)

Critical Points: (x_0,y_0) is a critical point on $z=f(x,y)$ if $f_x(x_0,y_0)=f_y(x_0,y_0)=0$ or either $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ DNE.

Mass and Moments for Surfaces and Solids

Mass	Moments	Moments of Inertia
Object Q with density function $\rho(x,y,z)$ $M=\iiint_Q \rho(x,y,z)\,dV$ Center of Mass: $\left(\bar{x}=\frac{M_{yz}}{m},\bar{y}=\frac{M_{xz}}{m},\bar{z}=\frac{M_{xy}}{m}\right)$ Area and Volume: $A=\iint_A 1\,dA \qquad V=\iiint_Q 1\,dV$	Two Dimensions $M_x=\iint_R y\rho(x,y)dA$ $M_y=\iint_R x\rho(x,y)dA$ Three Dimensions $M_{xy}=\iiint_Q z\rho(x,y,z)\,dV$ $M_{xz}=\iiint_Q y\rho(x,y,z)\,dV$ $M_{yz}=\iiint_Q x\rho(x,y,z)\,dV$	Two Dimensions $I_x=\iint_R y^2\rho(x,y)dA$ $I_y=\iint_R x^2\rho(x,y)dA$ Three Dimensions $I_x=\iiint_Q \left(y^2+z^2\right)\rho(x,y,z)\,dV$ $I_y=\iiint_Q \left(x^2+z^2\right)\rho(x,y,z)\,dV$ $I_z=\iiint_Q \left(x^2+y^2\right)\rho(x,y,z)\,dV$

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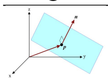
Common Shapes and Parameterizations

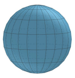
Conversion of Coordinates

Polar	Cylindrical	Spherical
$x = r\cos\theta$, $y = r\sin\theta$	$x = r\cos\theta$, $y = r\sin\theta$	$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$
$r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$	$r^2 = x^2 + y^2$, $z = z$, $\tan\theta = \frac{y}{x}$	$z = \rho\cos\phi$, $\rho^2 = x^2 + y^2 + z^2$
$dA = r\,dr\,d\theta$	$dV = r\,dz\,dr\,d\theta$	$dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

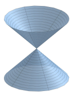
Plane

Image	Cartesian Form	Cylindrical Parameterization
	$Ax + By + Cz = D$ $\overrightarrow{N} = \langle A, B, C\rangle$	$x = r\cos\theta \quad y = r\sin\theta$ $z = D - Ar\cos\theta - Br\sin\theta \quad 0 < \theta < 2\pi$

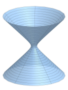
Ellipsoid

Image	Cartesian Form	Spherical Parameterization
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	$x = a\sin\phi\cos\theta \quad y = b\sin\phi\cos\theta \quad z = c\cos\phi$ $0 < \phi < \pi,\ 0 < \theta < 2\pi$

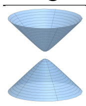
Elliptic Cone

Image	Cartesian Form	Cylindrical Parameterization
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	$x = ar\cos\theta \quad y = br\sin\theta \quad z = cr$ $0 \le r < \infty,\ 0 < \theta < 2\pi$

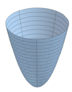
Hyperboloid of One Sheet

Image	Cartesian Form	Cylindrical Parameterization
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$x = a\sqrt{1+r^2}\cos\theta \quad y = b\sqrt{1+r^2}\sin\theta \quad z = cr$ $0 \le r < \infty,\ 0 < \theta < 2\pi$

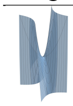
Hyperboloid of Two Sheets

Image	Cartesian Form	Cylindrical Parameterization
	$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sqrt{r^2-1}\cos\theta \quad y = b\sqrt{r^2-1}\sin\theta \quad z = cr$ $0 \le r < \infty,\ 0 < \theta < 2\pi$

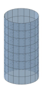
Elliptic Paraboloid

Image	Cartesian Form	Cylindrical Parameterization
	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	$x = \sqrt{ar}\cos\theta \quad y = \sqrt{br}\sin\theta \quad z = r$ $0 \le r < \infty,\ 0 < \theta < 2\pi$

Hyperbolic Paraboloid

Image	Cartesian Form	Cylindrical Parameterization
	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	$x = ar\cos\theta \quad y = -br\sin\theta \quad z = r^2(\cos^2\theta - \sin^2\theta)$ $0 \le r < \infty,\ 0 < \theta < 2\pi$

Cylinder

Image	Cartesian Form	Cylindrical Parameterization	Spherical Parameterization
	$(x-h)^2 + (y-k)^2 = r^2$	$x = h + \cos\theta \quad y = k + \sin\theta$ $z = r \quad r$ is constant, $0 < \theta < 2\pi$	(centered at origin without h and k) $\rho = r\csc\phi \quad x = r\cos\theta \quad y = r\sin\theta$ $z = r\cos\theta \quad 0 < \theta < 2\pi \quad 0 < \rho < \pi$