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POSITIVE SOLUTIONS TO A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}$

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Abstract. We discuss the existence of positive solutions of perturbation to a class of quasilinear elliptic equations on $\mathbb{R}$.

1. **Introduction.** This Note is concerned with solutions of perturbations to the following problem on the real line $\mathbb{R}$

$$\begin{cases} -u'' + u - k(u^2)''u = u^p, \\ u \in H^1(\mathbb{R}) \quad u > 0, \end{cases} \tag{1}$$

where $k \in \mathbb{R}$ and $p > 1$. Consider

$$\begin{cases} -u'' + (1 + \varepsilon a(x))\, u - k\,(1 + \varepsilon b(x))\,(u^2)''u = (1 + \varepsilon c(x))\, u^p, \\ u \in H^1(\mathbb{R}) \quad u > 0, \end{cases} \tag{2}$$

where a, b, c are assumed to be real valued functions belonging to the class S,

$$\begin{aligned} S \quad = \quad & \{h(x) = h_1(x) + h_2(x) : h_1 \in L^r(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ & \text{for some } r \in [1, \infty),\ h_2 \in L^\infty(\mathbb{R}),\ \lim_{|x|\to\infty} h_2(x) = 0\}. \end{aligned}$$

Our main existence result is the following

Theorem 1.1. *There is $k_0 > 0$ such that for $k > -k_0$ and $a, b, c \in S$, equation* (2) *has a solution provided $|\varepsilon|$ is sufficiently small.*

Solutions of (2) will be found as critical points of a functional I_ε of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u), \qquad u \in H^1(\mathbb{R}), \tag{3}$$

where

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}} \left(|u'|^2 + u^2\right) dx + k\int_{\mathbb{R}} u^2|u'|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1} dx \tag{4}$$

and G is the perturbation

$$G(u) = \frac{1}{2}\int_{\mathbb{R}} \left(a(x)u^2\right) dx + k\int_{\mathbb{R}} b(x)u^2|u'|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} c(x)|u|^{p+1} dx. \tag{5}$$

Critical points of I_ε will be found using the abstract perturbation procedure introduced in [1, 2]. Due to the translation invariance, the critical points of I_0 appear as manifolds. We first show that I_0 has a unique positive critical point z, up to translations, which is nondegenerate, in the sense that $I_0''(z)$ has a one dimensional

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kernel (given by the translations). Then, for $|\varepsilon|$ small, I_ε will be shown to have a critical point which is close to the critical manifold of I_0.

We first consider the equation

$$\begin{cases} -u'' + V(x)u - k(u^2)''u = \lambda u^p, \\ u \in H^1(\mathbb{R}) \quad u > 0. \end{cases} \tag{6}$$

When $k = 0$ and $\lambda > 0$ is fixed, solutions of (6) are nothing but the standing waves of the corresponding nonlinear Schrödinger equation, whose existence has been widely investigated, see [8] and e.g. [3, 5].

Equation (6) with $k > 0$ arises in various fields of physics, like the theory of superfluids or in dissipative quantum mechanics, see e.g. [9, 14, 15]. Equations with more general dissipative term arise in plasma physics, fluid mechanics, in the theory of Heisenberg ferromagnets, etc. For further physical motivations and a more complete list of references dealing with applications, we refer to [10, 16] and to their bibliography.

In [16] equation (6) has been studied. For $k > 0$, for several classes of potentials $V(x)$ and certain ranges of $p > 1$, ground state solutions were constructed as minimizers of a constrained minimization problem, with λ being the Lagrange multiplier. An Orlicz space approach was used in [13] to obtain ground state solutions for prescribed $\lambda > 0$. The concentration compactness principle was used in these papers to overcome the lack of compactness.

The results stated in Theorem 1.1 are new. However, even if we take $b \equiv c \equiv 0$, when (2) becomes (6) with $V(x) = 1 + \varepsilon a(x)$, the perturbative approach allows us to construct solutions for a range of k, V and p not covered by [13, 16], including cases when the ground state solution may not exist. In particular, it is worth mentioning that we can handle some cases in which k is negative. See also Remark (i) at the end of Section 3. We also give a different approach for obtaining ground state solutions of (6) for prescribed $\lambda > 0$. The results we find are similar to those of [13] but we use an approach which is technically simpler than that in [13].

In Section 2 we first study the ground state solutions. Then Section 3 is devoted to proving Theorem 1.1.

2. **Ground state solutions.** We first deal with the problem

$$\begin{cases} -u'' + V(x)u - k(u^2)''u = \lambda u^p, \\ u \in H^1(\mathbb{R}) \quad u > 0. \end{cases}$$

We will consider $\lambda > 0$ as a fixed parameter. Since in the following the arguments are independent of such $\lambda > 0$, we will take $\lambda \equiv 1$ for simplicity of notation.

We assume always that

$(V0)$**:** $\inf_{\mathbb{R}} V(x) > 0$.

Let

$$X = \{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2 dx < +\infty\}$$

endowed with norm

$$\|u\|^2 = \int_{\mathbb{R}} \left(|u'|^2 + V(x)u^2\right) dx.$$

The corresponding scalar product in X will be denoted by $(\cdot,\cdot)$. Then solutions of (6) are positive critical points of

$$I(u)=\frac{1}{2}\|u\|^2+k|uu'|_2^2-\frac{1}{p+1}|u|_{p+1}^{p+1},$$

where we use $|u|_q$ to denote the $L^q(\mathbb{R})$ norm. Let

$$\gamma(u):=(I'(u),u)=\|u\|^2+4k|uu'|_2^2-|u|_{p+1}^{p+1}$$

and define the *Nehari manifold* by setting

$$M=\{u\in X\setminus\{0\}:\gamma(u)=0\}.$$

In the next two lemmas we collect the main properties of M that we need in the sequel.

Lemma 2.1. *Let $k>0$ and $p\geq 3$. Then $M\neq\emptyset$. Moreover, for any $u\in X\setminus\{0\}$ with $\gamma(u)<0$ there exists a unique $\theta_u\in]0,1[$ such that $\theta_u u\in M$.*

PROOF. Fix $u\in X\setminus\{0\}$. One has $\gamma(\theta u)=\theta^2 g(\theta)$, where $g(\theta)=\|u\|^2+4k\theta^2|uu'|_2^2-\theta^{p-1}|u|_{p+1}^{p+1}$. Then $g(0)=\|u\|^2>0$ and, for $\theta>0$ one has

$$g'(\theta)=8k\theta|uu'|_2^2-(p-1)\theta^{p-2}|u|_{p+1}^{p+1}. \tag{7}$$

Let us distinguish between the case $p>3$ and $p=3$. In the former, from (7) we find that $g'(\theta)<0$, resp. >0, for

$$\theta>\theta^*:=\left(\frac{8k|uu'|_2^2}{(p-1)|u|_{p+1}^{p+1}}\right)^{1/(p-3)},\qquad \text{resp } \theta<\theta^*,$$

and the lemma follows when $p>3$.

Next, let us consider the case $p=3$. Recalling that $\inf\{\int_{\mathbb{R}}|\phi'|^2dx:\phi\in H^1(\mathbb{R}),\ \int_{\mathbb{R}}\phi^2dx=1\}=0$, we infer that for all $\varepsilon>0$ there exists $\phi_\varepsilon\in H^1(\mathbb{R})$, with $\int_{\mathbb{R}}\phi_\varepsilon^2dx=1$ such that $\int_{\mathbb{R}}|(\phi_\varepsilon)'|^2dx=\varepsilon$. Taking $u_\varepsilon^2=\phi_\varepsilon$ we get:

$$|u_\varepsilon u_\varepsilon'|_2^2=\int_{\mathbb{R}}u_\varepsilon^2|u_\varepsilon'|^2dx=\frac{1}{4}\int_{\mathbb{R}}|(u_\varepsilon^2)'|^2dx=\frac{1}{4}\int_{\mathbb{R}}|\phi_\varepsilon'|^2dx=\frac{1}{4}\varepsilon,$$

while $|u_\varepsilon|_4^4=|\phi_\varepsilon|_2^2=1$. Corresponding to such u_ε, we find (recall that $p=3$):

$$g(\theta)=\|u_\varepsilon\|^2+\alpha_\varepsilon\theta^2,\qquad\text{where }\ \alpha_\varepsilon:=4k|u_\varepsilon u_\varepsilon'|_2^2-|u_\varepsilon|_4^4=k\varepsilon-1\ .$$

Choosing $\varepsilon<k$ we get that $\alpha_\varepsilon<0$ and the result follows. ■

Lemma 2.2. *Let $p\geq 3$. Then M is a manifold and (if $M\neq\emptyset$) one has:*

- *(i)***:** *there exists $\rho>0$ such that $\|u\|\geq\rho$, $\ \forall\, u\in M$.*
- *(ii)***:** *If $u^*\in M$ is a constrained critical point of I on M, then $I'(u^*)=0$ and u^* is a solution of* (6)*.*
- *(iii)***:** *$m:=\inf_{u\in M}I(u)\geq const.>0$.*

PROOF. From $\gamma(u)=0$ it follows that

$$4k|uu'|_2^2=|u|_{p+1}^{p+1}-\|u\|^2. \tag{8}$$

Then for $u\in M$ one has

$$(\gamma'(u),u)=2\|u\|^2+16k|uu'|_2^2-(p+1)|u|_{p+1}^{p+1}=-2\|u\|^2+(3-p)|u|_{p+1}^{p+1}<0. \tag{9}$$

In particular, M is a manifold. Furthermore, (8) also yields

$$\|u\|^2=-4k|uu'|_2^2+|u|_{p+1}^{p+1}\leq C\|u\|^{p+1}\quad(u\in M),$$

proving (i). In addition, for the u^* in the lemma, there exists $\mu \in \mathbb{R}$ such that $I'(u^*) = \mu\gamma'(u^*)$. Taking the scalar product, one finds $(I'(u^*), u^*) = \mu(\gamma'(u^*), u^*)$. Since $u^* \in M$ then $(I'(u^*), u^*) = 0$ while $(\gamma'(u^*), u^*) < 0$, according to (9). Thus $\mu = 0$, proving (ii).
Using again (8), we infer that

$$I_{|M}(u) = \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)|u|_{p+1}^{p+1}. \tag{10}$$

Therefore, if $p \geq 3$ and taking into account (i), we deduce that $m \geq const. > 0$. ■

For future references let us recall the following lemma from [16].

Lemma 2.3. *Let $u_n \rightharpoonup u$ in X and set $v_n = u_n - u$. Then*

$$\liminf |u_n u_n'|_2^2 \geq \liminf |v_n v_n'|_2^2 + |uu'|_2^2\ .$$

We are now in position to prove the existence of ground states for (6).

Theorem 2.4. *Let $(V0)$ hold and suppose that either V satisfies*

$(V1)$**:** $\lim_{|x|\to\infty} V(x) = +\infty$*;*

or is bounded and satisfies

$(V2)$**:** *V is periodic in x;*

$(V3)$**:** *$\lim_{|x|\to\infty} V(x)$ exists and denoted by $\overline{V}$ such a limit, one has that $\overline{V} = \sup_{\mathbb{R}} V(x)$.*

Then for $k > 0$ and $p \geq 3$, m is always achieved at some $u \in M$, which is a solution of (6).

Remarks. (i) When V satisfies $(V0)$, $(V3)$ and is of the form $V(x) = 1 + \varepsilon a(x)$ we could also apply Theorem 1.1, with $b(x) \equiv c(x) \equiv 0$. However, let us point out that $(V3)$ implies that $\varepsilon a(x) < 0$. On the contrary, no such restriction is required in Theorem 1.1 where, in addition, we allow k to be negative and $p > 1$.

(ii) There are cases not covered by Theorem 2.4 in which M is not empty and $I_{|M}$ is bounded below. For example, if $k < 0$ and $1 < p < 3$ one has that $g(\theta u) = \theta^2\left(\|u\|^2 + 4k|uu'|_2^2\theta^2 - |u|_{p+1}^{p+1}\theta^{p-1}\right)$ and this implies that $\forall\, u \in X \setminus \{0\}$, $\exists\, \theta_u$ such that $\theta_u u \in M$. Moreover,

$$I(u) = \frac{p-1}{2(p+1)}\|u\|^2 + k\frac{p-3}{p+1}|uu'|_2^2 > 0, \quad \forall\, u \in M.$$

Although we do not know in the present generality if the $\inf_M I(u)$ is achieved, we will show later on that a ground state exists provided $k > -k_0$, $p > 1$, $V(x) = 1 + \varepsilon a(x)$, $a \in S$ and $\varepsilon a(x) < 0$, see the Remark at the end of Section 3.

(iii) The proof below will make it clear that assumption $(V1)$ can be replaced by any other condition that assures the compact embedding from X into $L^{p+1}(\mathbb{R})$.

Proof. Let $u_n \in M$ be a minimizing sequence. Clearly u_n is bounded and we can assume that $u_n \rightharpoonup u$ in X. First, some preliminary remarks are in order. Since $u_n \in M$, then $|u_n|_{p+1}^{p+1} = \|u_n\|^2 + 4k|u_n u_n'|_2^2 \geq \|u_n\|^2$ and therefore (i) of lemma 2.1 yields

$$\lim |u_n|_{p+1}^{p+1} > 0. \tag{11}$$

Furthermore, setting as in the preceding Lemma $v_n = u_n - u$, we claim that

$$\gamma(u) + \liminf \gamma(v_n) \le 0, \tag{12}$$

so that either $\gamma(u) \le 0$ or $\liminf \gamma(v_n) \le 0$ (or both). To prove (12) we argue as follows. Since $u_n \in M$, then $0 = \gamma(u_n) = \|u_n\|^2 + 4k|u_n u_n'|_2^2 - |u_n|_{p+1}^{p+1}$. By a well known result by Brezis and Lieb [7],

$$|u_n|_{p+1}^{p+1} = |v_n|_{p+1}^{p+1} + |u|_{p+1}^{p+1} + o(1).$$

Also, $\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1)$. Using these equations as well as Lemma 2.3 we infer:

$$\begin{aligned} 0 &= \|u_n\|^2 + 4k|u_n u_n'|_2^2 - |u_n|_{p+1}^{p+1} \\ &= \|v_n\|^2 + \|u\|^2 + 4k|u_n u_n'|_2^2 - |v_n|_{p+1}^{p+1} - |u|_{p+1}^{p+1} + o(1) \\ &\ge \liminf \gamma(v_n) + \gamma(u), \end{aligned}$$

proving (12).
We now distinguish among the three assumptions made on V.

(a) If ($V1$) holds then X is compacly embedded in L^{p+1}, see [6], and hence $u_n \to u$ strongly in L^{p+1} (up to a subsequence). From (11) it follows that $u \ne 0$. Moreover, one has

$$\gamma(v_n) = \|v_n\|^2 + 4k|v_n v_n'|_2^2 - |v_n|_{p+1}^{p+1} \ge \|v_n\|^2 - |v_n|_{p+1}^{p+1} = \|v_n\|^2 + o(1).$$

If $\lim \|v_n\|^2 \ne 0$ (otherwise we have done) it follows that $\liminf \gamma(v_n) > 0$. Then from (12) it follows that $\gamma(u) < 0$. So, $u \ne 0$ and Lemma 2.1 imply that there exists $\theta = \theta_u \in]0,1[$ such that $\theta u \in M$. Since u_n is a minimizing sequence and using (10), we deduce

$$\begin{aligned} m + o(1) &= I(u_n) = \frac{1}{4}\|u_n\|^2 + \frac{p-3}{4(p+1)}|u_n|_{p+1}^{p+1} \\ &\ge \frac{1}{4}\|u\|^2 + \frac{p-3}{4(p+1)}|u|_{p+1}^{p+1} + o(1) \\ &= \frac{1}{4}\,\theta^{-2}\|\theta u\|^2 + \frac{p-3}{4(p+1)}\,\theta^{-(p+1)}|\theta u|_{p+1}^{p+1} + o(1). \end{aligned}$$

Since $0 < \theta < 1$ it follows that

$$m > \frac{1}{4}\|\theta u\|^2 + \frac{p-3}{4(p+1)}|\theta u|_{p+1}^{p+1} = I(\theta u),$$

a contradiction because $\theta u \in M$. Thus, $v_n \to 0$ strongly in X, $u \in M$ and u is a minimizer of $I_{|M}$.

(b) Suppose that ($V2$) holds. From (11) and by the Concentration Compactness Principle [11], we infer that there exist $r > 0$ and $y_n \in \mathbb{R}$ such that

$$\liminf_{n\to\infty} \int_{y_n - r}^{y_n + r} |u_n|^{p+1} dx > 0. \tag{13}$$

We may assume y_n are integer multipliers of the period of V. Then the sequence $\tilde{u}_n(x) = u_n(x + y_n)$ is also a minimizing sequence because $I(\tilde{u}_n) = I(u_n)$ in view of the periodicity of V. Without loss of generality, assume that $y_n = 0$ and hence (13) implies that $u \ne 0$. If $\gamma(u) = 0$ it is easy to see that $\|u_n - u\| \to 0$ and $I(u) = m$. Otherwise suppose that $\gamma(u) \ne 0$. If $\gamma(u) < 0$ we argue as in the preceding point

(a). It remains to consider the case in which, up to a subsequence, $\lim \gamma(v_n) < 0$. Then, for n large, there exist $\theta_n = \theta_{v_n} \in]0,1[$ such that $\theta_n v_n \in M$. Furthermore one has that $\limsup \theta_n < 1$, otherwise, along a subsequence, $\theta_n \to 1$ and hence $\gamma(v_n) = \gamma(\theta_n v_n) + o(1) = o(1)$, a contradiction. As before, we find

$$m + o(1) = I(u_n) \geq \frac{1}{4}\, \theta_n^{-2} \|\theta_n v_n\|^2 + \frac{p-3}{4(p+1)}\, \theta_n^{-(p+1)} |\theta_n v_n|_{p+1}^{p+1} + o(1).$$

Therefore one has again $m > I(\theta_n v_n)$, for n large, a contradiction. (c) Suppose that $(V3)$ holds. First consider

$$\overline{I}(u) = \frac{1}{2}\int_{\mathbb{R}} \left(|u'|^2 + \overline{V}u^2\right) dx + k\int_{\mathbb{R}} u^2|u'|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} |u|^p dx.$$

By the result proved in point (b), $\overline{I}$ has a critical point $\overline{u}$ which is a minimizer of $\overline{I}$ on $\overline{M} := \{u \in H^1(\mathbb{R}) : (\overline{I}'(u), u) = 0\}$. It is readily proved that $\overline{u} > 0$ on $\mathbb{R}$. Also, we use the notation $\overline{m}$, $\overline{\gamma}$ or $\overline{\theta}$ for this problem, with obvious meaning. If V is not constant (otherwise (b) applies), one easily gets $\gamma(u) < \overline{\gamma}(u)$ for all u. In particular, $\gamma(\overline{u}) < \overline{\gamma}(\overline{u}) = 0$. Then $\theta(\overline{u}) \in]0,1[$, and

$$m \leq I(\theta(\overline{u})\overline{u}) < \overline{I}(\overline{u}) = \overline{m}. \tag{14}$$

Let us take again a minimizing sequence for I and let us show that $u \neq 0$. Otherwise, from $I(u_n) - \overline{I}(u_n) = \int_{\mathbb{R}} (V(x) - \overline{V}) u_n^2 dx$ and since $\lim_{|x|\to\infty} V(x) = \overline{V}$ we get

$$\lim \overline{I}(u_n) = \lim I(u_n) = m.$$

For the same reason, we also have $0 = \gamma(u_n) = \overline{\gamma}(u_n) + o(1)$. Let $\overline{\theta}_n$ be such that $\overline{\theta}_n u_n \in \overline{M}$. From $\overline{\gamma}(\overline{\theta}_n u_n) = 0$ we infer for $p > 3$:

$$\overline{\theta}_n^2 \left(\overline{\theta}_n^{p-3} |u_n|_{p+1}^{p+1} - 4k|u_n u_n'|_2^2\right) = \|u_n\|^2 + o(1).$$

Since $\|u_n\|$ and $|u_n u_n'|_2$ are bounded and using (11) we deuce that $\overline{\theta}_n$ is bounded. The case $p = 3$ can be handled similarly. Then, using also the property of the Nehari manifold that $I(\lambda u) \leq I(u)$ for all $u \in M$ and all $\lambda > 0$, we find

$$\overline{m} \leq \overline{I}(\overline{\theta}_n u_n) = I(\overline{\theta}_n u_n) + o(1) \leq I(u_n) + o(1) = m + o(1),$$

a contradiction with (14). This proves that $u \neq 0$. Now we proceed as before: one has that $m + o(1) = I(u_n) = \overline{I}(v_n) + I(u) + o(1)$ and since $0 = \gamma(u_n) \geq \gamma(u) + \liminf \overline{\gamma}(v_n)$, we get that either $\gamma(u) \leq 0$ or $\liminf \overline{\gamma}(v_n) \leq 0$, or both. Then the arguments carried out above yield a contradiction, proving that $\|u_n - u\| \to 0$, $u \in M$ and m is achieved there.

In any case, we have shown that $u \in M$ is a minimizer for $I_{|M}$ provided that one among $(V1), (V2)$ or $(V3)$ holds. Using (ii) of Lemma 2.2 it follows that u solves the equation in (6). It remains to show that $u > 0$ and this follows as in [16]. Since $|u_n|$ is also a minimizer, one infers that $u \geq 0$. Note that the equation in (6) can be written in the form

$$-(1 + 2ku^2)u'' = \left(-V(x) + 2k(u')^2 + |u|^{p-1}\right) u. \tag{15}$$

If $u(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then $u'(x_0) = 0$ and by the uniqueness of the Cauchy problem, (15) implies that $u \equiv 0$, a contradiction. This shows that $u > 0$ and completes the proof. ■

The autonomous problem. In the remaining part of this section, we consider the autonomous problem (1). Setting $v = u'$, (1) is equivalent to the Hamiltonian system on $\mathbb{R}^2$

$$\begin{cases} u' &= v \\ v' &= (u - 2kv^2u - |u|^{p-1}u)(1 + 2ku^2)^{-1}. \end{cases} \tag{16}$$

The energy

$$E(u, v) = (1 + 2ku^2)\frac{v^2}{2} - \frac{u^2}{2} + \frac{|u|^{p+1}}{p+1}$$

is constant along the trajectories of (16) and a standard phase plane analysis shows that there exists $k_0 > 0$ such that $(0, 0)$ is a hyperbolic equilibrium provided $k > -k_0$ and $p > 1$. Moreover, for such k, p, the equation $E(u, v) = 0$ defines an isolated orbit $(z(x), z'(x))$ homoclinic to $(0, 0)$, with $z(x) > 0$ for all $x \in \mathbb{R}$. By homoclinic to $(0, 0)$ we mean, as usual, that $z(x) \to 0$ and $z'(x) \to 0$ as $|x| \to \infty$. In addition, z is even in x and z, z', z'' all tend to zero exponentially as $|x| \to \infty$. This $z(x)$ is the unique even positive solution of (1). Of course, all translations $z_\xi(x) = z(x+\xi)$, $\xi \in \mathbb{R}$, are also solutions of (1) and form a one dimensional manifold $Z = \{z_\xi : \xi \in \mathbb{R}\}$. By the uniqueness of the Cauchy problem, it is easy to see that the dimension of $\mathrm{Ker}[I_0''(z_\xi)]$ equals the dimension of the tangent space to the homoclinic trajectory $(z(x), z'(x))$ in $\mathbb{R}^2$. This means that $\dim(\mathrm{Ker}[I_0''(z_\xi)]) = 1$. Since, of course, the tangent space at z_ξ to Z, $T_{z_\xi}Z$, is a subset of $\mathrm{Ker}[I_0''(z_\xi)]$, by dimensionality one infers that Z is *nondegenerate* in the sense that

$$T_{z_\xi}Z = \mathrm{Ker}[I_0''(z_\xi)], \qquad \forall\, z_\xi \in Z. \tag{ND}$$

In conclusion, we have shown:

Proposition 2.5. *There exits $k_0 > 0$ such that for all $k > -k_0$ and all $p > 1$, equation* (1) *has a one dimensional manifold Z of critical points such that* (ND) *holds.*

Remark. Let us explicitly point out that, for any $k > -k_0$ there is $c_k > 0$ such that $(1 + 2kz^2(x)) \geq c_k$ for all $x \in \mathbb{R}$.

3. **Proof of theorem 1.1.** Solutions of (2) will be found as critical points of the functional I_ε defined in (3). This will be done by means of the perturbation method discussed in [1, 2], that we briefly recall for the reader convenience. For more details we refer to the forementioned papers, as well as [4].

Here we merely take $X = H^1(\mathbb{R})$, with the standard norm

$$\|u\|^2 = \int_{\mathbb{R}} \left(|u'|^2 + u^2\right) dx.$$

Moreover it is always understood that $k > -k_0$. We look for critical points of I_ε of the form $z_\xi + w$, where $z_\xi \in Z$ and $w \in (T_{z_\xi}Z)^\perp$, where Z has been introduced in the last part of Section 2. Let P_ξ denote the projection of X to $(T_{z_\xi}Z)^\perp$. We first solve the auxiliary equation $P_\xi I_\varepsilon'(z_\xi + w) = 0$, namely

$$P_\xi[I_0'(z_\xi + w) + \varepsilon G'(z_\xi + w)] = 0. \tag{17}$$

Since $I_0'(z_\xi) = 0$ one finds $I_0'(z_\xi + w) = I_0''(z_\xi) + R(w)$, where $R(w) = o(\|w\|)$ as $\|w\| \to 0$. Then (17) is equivalent to $P_\xi I_0''(z_\xi) + \varepsilon P_\xi G'(z_\xi + w) + P_\xi R(w) = 0$. The following Lemma holds

Lemma 3.1. *Let $L_\xi := P_\xi I_0''(z_\xi)$. There exists $c > 0$ such that for all $\xi \in \mathbb{R}$*

$$|(L_\xi w, w)| \geq c\|w\|^2, \qquad \forall\, w \in (T_{z_\xi} Z)^\perp. \tag{18}$$

Proof. It suffices to consider the linearized operator

$$I_0''(z_\xi) : v \mapsto -\left((1+2kz_\xi^2)v'\right)' + \left(1 - 4kz_\xi z_\xi'' - 2k(z_\xi')^2 - pz_\xi^{p-1}\right)v,$$

and to remark that, according to the Sturmian theory, it has a sequence of simple eigenvalue λ_i. The first one, λ_1 is negative (this is due to the fact that z_ξ is a mountain-pass critical point of I_0); moreover, by the nondegeneracy condition (ND) it follows that $\lambda_2 = 0 < \lambda_3 < \ldots$. Finally it is easy to check that the inequalities $\lambda_1 < 0 < \lambda_3$ are uniform with respect to $\xi \in \mathbb{R}$. ■

From the preceding Lemma it follows that (17) can be written as

$$w = (-L_\xi)^{-1}[\varepsilon P_\xi G'(z_\xi + w) + P_\xi R(w)].$$

Since the right hand side is a contraction, one can see that, for $|\varepsilon|$ is sufficiently small, there exists $\rho(\varepsilon)$, with $\rho(\varepsilon) \to 0$ as $|\varepsilon| \to 0$, such that (17) has a unique solution $w_{\xi,\varepsilon}$ of class C^1 with respect to ξ satisfying

$$\|w_{\xi,\varepsilon}\| \leq \rho(\varepsilon). \tag{19}$$

Once the auxiliary equation is solved, we consider the one dimensional functional

$$\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_{\xi,\varepsilon}).$$

According to the results of [1, 2],

- **$(*)$:** if $\xi_0 \in \mathbb{R}$ is such that $\Phi_\varepsilon'(\xi_0) = 0$ then $z_{\xi_0} + w_{\xi_0,\varepsilon}$ is a critical point of I_ε and hence a solution of (2).

We shall show that

$$\lim_{|\xi|\to\infty} \Phi_\varepsilon(\xi) = const. \tag{20}$$

and this will imply, in view of $(*)$, that (2) has a solution. Thus the rest of this section is devoted to show that (20) holds. For simplicity, we consider $b(x) = c(x) \equiv 0$, so that

$$G(u) = \frac{1}{2}\int_{\mathbb{R}} a(x)u^2 dx.$$

However, from the proof it will be clear that the result holds for non zero b, c, too. The procedure we follow is similar to that used in [4], but the presence of the term $(u^2)''u$ requires different arguments here. In order to show that (20) holds we shall prove that

Proposition 3.2. *For all $|\varepsilon|$ small, $\lim_{|\xi|\to\infty}\|w_{\xi,\varepsilon}\| = 0$.*

The proof of this Proposition is postponed after several preliminary Lemmas. In the sequel, for brevity, we will write w_ξ instead of $w_{\xi,\varepsilon}$. First, of all, since w_ξ is uniformly bounded in $\xi \in \mathbb{R}$, then $w_\xi \rightharpoonup w_\infty (= w_{\infty,\varepsilon})$, along a sequence $|\xi_n| \to \infty$. Such a w_ξ satisfies the auxiliary equation (17). A direct computation yields

$$\begin{aligned}
I_0''(z_\xi)[w] &= -w'' + w - 2kwz_\xi(z_\xi)'' - 2kz_\xi^2 w'' - kw(z_\xi^2)'' \\
&\quad -4kz_\xi(z_\xi)'w' - pz_\xi^{p-1}w, \\
G'(z_\xi + w) &= a(x)(z_\xi + w), \\
R(w) &= z_\xi^p - (z_\xi + w)^p - k\left[(w^2)''(z_\xi + w) + 2(z_\xi w)''w\right] + pz_\xi^{p-1}w.
\end{aligned}$$

Therefore, applying the projection P_ξ we find there exists $\alpha_\xi \in \mathbb{R}$ such that w_ξ satisfies the following differential equation

$$\left\{\begin{array}{l} -w_\xi'' + w_\xi - 2kw_\xi z_\xi (z_\xi)'' - 2kz_\xi^2 (w_\xi)'' - kw_\xi (z_\xi^2)'' \\ \quad -4kz_\xi (z_\xi)' w_\xi' + \varepsilon a(x)(z_\xi + w_\xi) + z_\xi^p \\ \quad -(z_\xi + w_\xi)^p - k\left[(w_\xi^2)''(z_\xi + w_\xi) + 2(z_\xi w_\xi)'' w_\xi\right] + \alpha_\xi z_\xi' = 0. \end{array}\right. \tag{21}$$

Here we have used the fact that $T_{z_\xi} Z$ is spanned by z_ξ'. Multiplying (21) by w_∞, integrating on $\mathbb{R}$ and taking into account that $w_\infty \perp T_{z_\xi} Z$, we get

$$\left\{\begin{array}{l} \int_{\mathbb{R}} (w_\xi' w_\infty' + w_\xi w_\infty) dx + k \int_{\mathbb{R}} (w_\xi^2)' (w_\xi w_\infty)' dx \\ \quad +\varepsilon \int_{\mathbb{R}} a(x)(z_\xi + w_\xi) w_\infty dx + \int_{\mathbb{R}} \left(z_\xi^p - (z_\xi + w_\xi)^p)\right) w_\infty dx \\ \quad - \int_{\mathbb{R}} \ell_\xi w_\infty dx = 0, \end{array}\right. \tag{22}$$

where

$$\ell_\xi = 2kw_\xi z_\xi (z_\xi)'' + 2kz_\xi^2 (w_\xi)'' + kw_\xi (z_\xi^2)'' + 4kz_\xi (z_\xi)'(w_\xi)' + k(w_\xi^2)'' z_\xi + 2k(z_\xi w_\xi)'' w_\xi.$$

We need to estimate the above integrals. This will be done in the next three lemmas.

Lemma 3.3.

(*i*)**:** $\lim_{|\xi|\to\infty} \int_{\mathbb{R}} (w_\xi' w_\infty' + w_\xi w_\infty) dx = \|w_\infty\|^2$;

(*ii*)**:** $\int_{\mathbb{R}} (w_\xi^2)'(w_\xi w_\infty)' dx = \int_{\mathbb{R}} |(w_\infty^2)'|^2 dx + 2\int_{\mathbb{R}} |(w_\xi - w_\infty)'|^2 w_\infty^2 dx + o(1)$, *as* $|\xi| \to \infty$.

PROOF. (*i*) follows immediately, because $\lim \int_{\mathbb{R}} (w_\xi' w_\infty' + w_\xi w_\infty) dx = (w_\xi, w_\infty) \to \|w_\infty\|^2$.

To prove (*ii*) we recall that $w_\xi w_\infty \rightharpoonup w_\infty^2$ in X as $|\xi| \to \infty$ yielding:

$$\left\{\begin{array}{l} \int_{\mathbb{R}} (w_\xi^2)'(w_\xi w_\infty)' dx = \int_{\mathbb{R}} (w_\xi^2 - w_\infty^2 + w_\infty^2)'(w_\xi w_\infty)' dx \\ \quad = \int_{\mathbb{R}} \left[(w_\xi - w_\infty)'(w_\xi + w_\infty) + (w_\xi - w_\infty)(w_\xi + w_\infty)'\right](w_\xi w_\infty)' dx \\ \quad + \int_{\mathbb{R}} |(w_\infty^2)'|^2 dx + o(1). \end{array}\right. \tag{23}$$

Moreover

$$\int_{\mathbb{R}} (w_\xi - w_\infty)'(w_\xi + w_\infty)(w_\xi w_\infty)' dx =$$
$$\int_{\mathbb{R}} \left[(w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\xi w_\infty)' + 2w_\infty (w_\xi - w_\infty)'(w_\xi w_\infty)'\right] dx.$$

But

$$\begin{aligned} \left|\int_{\mathbb{R}} (w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\xi w_\infty)' dx\right| &= \\ &= \left|\int_{|x|<R} (w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\xi w_\infty)' dx\right| \\ &+ \left|\int_{|x|>R} (w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\xi w_\infty)' dx\right| \\ &\le C\|w_\xi - w_\infty\|_{L^\infty(-R,R)} + C\int_{|x|>R} w_\infty^2 dx + C\int_{|x|>R} (w_\infty')^2 dx. \end{aligned}$$

Thus
$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}}(w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\xi w_\infty)'dx = 0.$$
Furthermore
$$\begin{aligned}
&\int_{\mathbb{R}} w_\infty(w_\xi - w_\infty)'(w_\xi w_\infty)'dx = \\
&\quad = \int_{\mathbb{R}} w_\infty(w_\xi - w_\infty)'((w_\xi - w_\infty)w_\infty)'dx + \int_{\mathbb{R}} w_\infty(w_\xi - w_\infty)'(w_\infty^2)'dx \\
&\quad = \int_{\mathbb{R}} |(w_\xi - w_\infty)'|^2 w_\infty^2 dx + \int_{\mathbb{R}} w_\infty(w_\xi - w_\infty)'(w_\xi - w_\infty)(w_\infty)'dx \\
&\quad = \int_{\mathbb{R}} |(w_\xi - w_\infty)'|^2 w_\infty^2 dx + o(1).
\end{aligned}$$
The last two equations imply
$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}}(w_\xi - w_\infty)'(w_\xi + w_\infty)(w_\xi w_\infty)'dx = 0. \tag{24}$$
Similarly
$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}}(w_\xi - w_\infty)(w_\xi + w_\infty)'(w_\xi w_\infty)'dx = 0.$$
Substituting this and (24) into (23), (ii) follows. ■

Lemma 3.4.

(i)**:** $\int_{\mathbb{R}}\left[z_\xi^p - (z_\xi + w_\xi)^p)\right] w_\infty dx = \int_{\mathbb{R}} |w_\infty|^p dx + o(1),$ *as* $|\xi| \to \infty$;

(ii)**:** $\lim_{|\xi|\to\infty}\int_{\mathbb{R}} a(x) z_\xi w_\infty dx = 0$

Proof. To prove (i) it suffices to remark that, obviously, $\int_{\mathbb{R}} z_\xi^p w_\infty dx = o(1)$ as $|\xi| \to \infty$ and that $z_\xi + w_\xi \rightharpoonup w_\infty$ as well as $(z_\xi + w_\xi)^p \rightharpoonup (w_\infty)^p$.

As for (ii), let $a = a_1 + a_2$ with $a_1 \in L^r(\mathbb{R})$ and $a_2 \in L^\infty(\mathbb{R})$ and set, for $i = 1, 2$ and $R > 0$, $A_i(R) = \int_{|x|<R} |a_i(x) z_\xi w_\infty| dx$ and $B_i(R) = \int_{|x|>R} |a_i(x) z_\xi w_\infty| dx$. One has ($r'$ denotes the conjugate exponent of r):
$$\begin{aligned}
A_1(R) &\le \max_{|x|<R} |z_\xi(x)| \left(\int_{|x|<R} |a_1(x)|^r dx\right)^{1/r} \left(\int_{|x|<R} |w_\infty(x)|^{r'} dx\right)^{1/r'} \\
&\le C_1 \max_{|x|<R} |z_\xi(x)|, \\
B_1(R) &\le C_2 \left(\int_{|x|>R} |a_1(x)|^r dx\right)^{1/r} \left(\int_{|x|>R} |z_\xi(x)|^{r'} dx\right)^{1/r'} \\
&\le C_3 \left(\int_{|x|>R} |z_\xi(x)|^r dx\right)^{1/r}.
\end{aligned}$$
Given $\eta > 0$, choose $R > 0$ such that $B_1(R) < \eta$. Since $\max_{|x|<R} |z_\xi(x)| \to 0$ as $|\xi| \to \infty$, we deduce that $A_1(R) + B_1(R) \le \eta + o(1)$ as $|\xi| \to \infty$. Moreover
$$A_2(R) \le C_4 \max_{|x|<R} |z_\xi(x)|, \quad B_2(R) \le C_5 \int_{|x|>R} |z_\xi(x)| dx$$
and, as before, one finds that $A_2(R) + B_2(R) \le \eta + o(1)$ as $|\xi| \to \infty$. Then (ii) follows. ■

Lemma 3.5. *There results:*

$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}} \ell_\xi w_\infty dx = 0.$$

Proof. Note that every term of $\ell_\xi w_\infty$ contains z_ξ or z'_ξ and also w_∞ or w'_∞. Then we write each term on $(-R,R)$ and on $\mathbb{R}\setminus(-R,R)$. In $(-R,R)$ we use that z_ξ and z'_ξ are small for $|\xi|\to\infty$; on $\mathbb{R}\setminus(-R,R)$ we use that $\int_{|x|>R} w_\infty^2$ and $\int_{|x|>R}(w'_\infty)^2$ are small if R is large. ■

We are now in position to show

Lemma 3.6. *For ε small enough, one has that $w_{\xi,\varepsilon} \rightharpoonup 0$ in $H^1(\mathbb{R})$ as $|\xi|\to\infty$.*

Proof. We have to show that $w_\infty = 0$. Using Lemmas 3.3, 3.4 and 3.5 and passing to the limit in (22) as $|\xi|\to\infty$ we get

$$\begin{aligned}\|w_\infty\|^2 + \varepsilon\int_{\mathbb{R}} a(x)w_\infty^2 dx &= \int_{\mathbb{R}} |w_\infty|^{p+1}dx - k\int_{\mathbb{R}} |(w_\infty^2)'|^2 dx \\ &\quad -2k\lim\int_{\mathbb{R}} |(w_\xi - w_\infty)'|^2 w_\infty^2 dx.\end{aligned}$$

This implies, for some constants $C_i > 0$ independent of ξ and ε,

$$(1-\varepsilon C_1)\|w_\infty\|^2 \le C_2\,\|w_\infty\|^{p+1} + |k|\,C_3\,\|w_\infty\|^4 + 2|k|\,C_4\,\|w_\infty\|^2 \lim\|w_\xi - w_\infty\|^2.$$

From (19), we have that $\|w_\infty\| \le \rho(\varepsilon)$ and $\|w_\xi - w_\infty\| \le \rho(\varepsilon)$ with $\rho(\varepsilon)\to 0$ as $|\varepsilon|\to 0$. Then we may choose $|\varepsilon|$ small such that the right hand side of the preceding inequality is smaller or equal to $\|w_\infty\|^2/2$, while $(1-\varepsilon C_1) > 1/2$ and this implies $w_\infty = 0$. ■

We are now in position to prove Proposition 3.2. Multiplying (21) by w_ξ and integrating on $\mathbb{R}$, we get

$$\begin{cases}(L_\xi w_\xi, w_\xi) = & \varepsilon\int_{\mathbb{R}} a(x)(z_\xi + w_\xi)w_\xi dx \\ & +\int_{\mathbb{R}}\left[(z_\xi+w_\xi)^p - z_\xi^p - p|z_\xi|^{p-1}w_\xi\right]w_\xi dx \\ & +k\int_{\mathbb{R}}\left[(w_\xi^2)''z_\xi + (w_\xi^2)''w_\xi + 2(z_\xi w_\xi)''w_\xi\right]w_\xi dx.\end{cases} \tag{25}$$

Since $w_\xi \to 0$ in L^q_{loc} for any $q\ge 1$ and $a = a_1 + a_2$ with $a_1\in L^r$ and $a_2\in L^\infty$, then

$$\lim_{|\xi|\to\infty}\int_{|x|<R} a(x)w_\xi^2 dx = 0, \qquad \forall\, R>0. \tag{26}$$

Moreover, using that $a_2(x)\to 0$ as $|x|\to\infty$, then for any $\delta>0$ we choose $R>0$ such that $|a_2(x)|<\delta$ for all $|x|>R$. Then we get

$$\left|\int_{|x|>R} a(x)w_\xi^2 dx\right| \le C_1\int_{|x|>R} |a_1(x)|^r dx + C_2\,\delta.$$

This and (26) imply

$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}} a(x)w_\xi^2 dx = 0.$$

Similarly,

$$\lim_{|\xi|\to\infty}\int_{\mathbb{R}} a(x)z_\xi w_\xi dx = 0.$$

For the second term in (25) we find readily

$$\left|\int_{\mathbb{R}}[(z_\xi+w_\xi)^p - z_\xi^p - p|z_\xi|^{p-1}w_\xi]w_\xi dx\right| \le C\|w_\xi\|^{p+1}.$$

In the last part of (25) each term contains either three w_ξ and w'_ξ or four of them. Then

$$\left|\int_{\mathbb{R}}[(w_\xi^2)''z_\xi w_\xi + (w_\xi^2)''w_\xi^2 + 2(z_\xi w_\xi)''w_\xi^2]dx\right| \le C(\|w_\xi\|^3 + \|w_\xi\|^4).$$

Thus from (25) we get

$$C_1\|w_\xi\|^2 \le C_2\left(\|w_\xi\|^{p+1} + \|w_\xi\|^3 + \|w_\xi\|^4\right) + o(1).$$

Since $\|w_\xi\| \le \rho(\varepsilon)$, and $\rho(\varepsilon) \to 0$, we must have $\|w_\xi\| \to 0$ as $|\xi| \to \infty$. This completes the proof of proposition 3.2.

We are now ready to show that (20) holds, namely that

$$\lim_{|\xi|\to\infty} \Phi_\varepsilon(\xi) = const.$$

Below, of course, we take $|\varepsilon|$ small in such a way that all the preceding results hold true. With easy calculation we find:

$$\begin{aligned}\Phi_\varepsilon(\xi) &= I_0(z_\xi + w_\xi) + \frac{\varepsilon}{2}\int_{\mathbb{R}} a(x)(z_\xi+w_\xi)^2 dx\\ &= I_0(z) + o(\|w_\xi\|) + \frac{\varepsilon}{2}\int_{\mathbb{R}} a(x)(z_\xi+w_\xi)^2 dx.\end{aligned}$$

Here we have used that $I_0(z_\xi) \equiv I_0(z)$. Arguing as before one has that $\int_{\mathbb{R}} a(x)(z_\xi + w_\xi)^2 dx \to 0$ as $|\xi| \to \infty$. In addition, we know by Proposition 3.2 that $\|w_\xi\| \to 0$ as $|\xi| \to \infty$. Then we infer that $\lim_{|\xi|\to\infty}\Phi_\varepsilon(\xi) = I_0(z_\xi) \equiv const$, proving (20). Since $\lim_{|\xi|\to\infty}\Phi_\varepsilon(\xi) = const$, then Φ_ε has at least a stationary point ξ_0. Thus, according to $(*)$, $u_\varepsilon = z_{\xi_0} + w_{\xi_0,\varepsilon}$ is a solution of (2).

Let us finally show that for any fix $k > -k_0$, the solution u_ε is positive on $\mathbb{R}$, provided $|\varepsilon|$ is small enough. Given $R > 0$ let $\zeta_R > 0$ be the minimum of $z_{\xi_0}(x)$ on the interval $[-R, R]$. Recalling that $\|w_{\xi_0,\varepsilon}\| \le \rho(\varepsilon)$, and $\rho(\varepsilon) \to 0$, we find $\varepsilon_R > 0$ such that $u_\varepsilon(x) > \zeta_R/2 > 0$ for all $x \in [-R, R]$. By regularity, $u_\varepsilon \in C^1 \cap L^\infty$ and $u_\varepsilon(x) \to 0$ as $|x| \to \infty$. Then we can choose R such that $|u_\varepsilon(x)|^{p-1} < 1/2$ for all $|x| > R$ and all $|\varepsilon| < \varepsilon_R$. Now, suppose by contradiction that u_ε takes some negative values on $|x| > R$ and let x_1, $|x_1| > R$, a point where $u_\varepsilon(x_1) = \min\{u_\varepsilon(x) : |x| > R\} < 0$. We claim that, taking $|\varepsilon|$ possibly smaller,

$$-(1 + 2ku_\varepsilon^2(x_1))u''_\varepsilon(x_1) \le 0. \tag{27}$$

This is trivial if $k \ge 0$. When $-k_0 < k < 0$ we use the fact that $u_\varepsilon = z_{\xi_0} + w_{\xi_0,\varepsilon}$ and that $\|w_{\xi_0,\varepsilon}\|_{L^\infty} \to 0$ as $|\varepsilon| \to 0$. Using the Remark after Proposition 2.5, it follows that (27) holds for $k \in]-k_0, 0[$, provided $|\varepsilon|$ is sufficiently small. Next, from (15), namely

$$-(1 + 2ku^2)u'' = \left(-(1 + \varepsilon a(x)) + 2k(u')^2 + |u|^{p-1}\right)u,$$

we infer that

$$\left(-(1 + \varepsilon a(x_1)) + |u_\varepsilon(x_1)|^{p-1}\right)u_\varepsilon(x_1) = -(1 + 2ku_\varepsilon^2(x_1))u''_\varepsilon(x_1) \le 0.$$

On the other side, choose $\varepsilon_1 > 0$ in such a way that $(1 + \varepsilon a(x)) > 1/2$ for $|\varepsilon| < \varepsilon_1$. Then, taking $|\varepsilon| < \min\{\varepsilon_R, \varepsilon_1\}$ we get that $(1 + \varepsilon a(x_1)) > 1/2 > |u_\varepsilon(x_1)|^{p-1}$ and thus

$$\left(-(1 + \varepsilon a(x_1)) + |u_\varepsilon(x_1)|^{p-1}\right) u_\varepsilon(x_1) > 0,$$

a contradiction, proving that $u_\varepsilon > 0$. This completes the proof of Theorem 1.1.

Remarks. (i) The results of [2] would imply that the solution found above is a mountain-pass solution, provided $\varepsilon a(x) < 0$, $|\varepsilon|$ small. Therefore, for all $k > -k_0$ and all $p > 1$ Theorem 1.1 yields, for $|\varepsilon|$ small, the existence of a ground state when $\varepsilon a(x) < 0$. Let us point out that, setting $V = 1 + \varepsilon a(x)$ with $a \in S$, assumption $(V3)$ in Theorem 2.4 holds whenever $\varepsilon a(x) < 0$. Of course, unlike in Theorem 2.4, here we deal only with perturbed potentials but cover a broader range of k and p.

(ii) According to [1, 2] one finds that

$$\Phi_\varepsilon(\xi) = I_0(z) + \varepsilon\Gamma(\xi) + o(|\varepsilon|),$$

where

$$\Gamma(\xi) = \frac{1}{2}\int_{\mathbb{R}} a(x+\xi)z^2 dx + k\int_{\mathbb{R}} b(x+\xi)z^2(z')^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} c(x+\xi)z^{p+1} dx.$$

As in [4] one can show that $\Gamma(\xi) \to 0$ as $|\xi| \to \infty$ provided $a, b, c \in L^r \cap L^\infty$. If $\Gamma(\xi) \not\equiv 0$ then Γ has a maximum or minimum which gives rise to a positive solution of (2). In general, we do not know if $a, b, c \not\equiv 0$ implies that $\Gamma(\xi) \not\equiv 0$ and this is the reason why we have carried out the preceding arguments leading to show that Φ_ε has a stationary point provided $a, b, c \in S$. However, in some circumstances it is possible to work with Γ and this often permits to obtain sharper results. For instance, conditions on a, b, c can be imposed so that one can even obtain multiplicity results. We will shortly illustrate such a fact with a specific example. Let us assume that a, b, c are smooth and even. By a stright calculation one finds

$$\begin{aligned}
\Gamma(0) &= \underbrace{\frac{1}{2}\int_{\mathbb{R}} a(x)z^2 dx}_{A} + k\underbrace{\int_{\mathbb{R}} b(x)z^2(z')^2 dx}_{B} - \underbrace{\frac{1}{p+1}\int_{\mathbb{R}} c(x)z^{p+1} dx}_{C}, \\
\Gamma'(0) &= \frac{1}{2}\int_{\mathbb{R}} a'(x)z^2 dx + k\int_{\mathbb{R}} b'(x)z^2(z')^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} c'(x)z^{p+1} dx = 0, \\
\Gamma''(0) &= \underbrace{\frac{1}{2}\int_{\mathbb{R}} a''(x)z^2 dx}_{A''} + k\underbrace{\int_{\mathbb{R}} b''(x)z^2(z')^2 dx}_{B''} - \underbrace{\frac{1}{p+1}\int_{\mathbb{R}} c''(x)z^{p+1} dx}_{C''}.
\end{aligned}$$

Therefore $\xi = 0$ is a stationary point of Γ and if, e.g. $A + kB - C < 0$ and $A'' + kB'' - C'' \neq 0$, then $\xi = 0$ gives rise to a solution of (2). Furthermore, if $A'' + kB'' - C'' < 0$ the function Γ achieves its global minimum at some $\xi^* \neq 0$ and such ξ^* gives rise to a second solution of (2) $u^*_\varepsilon \simeq z_{\xi^*} + w_{\xi^*}$, which is asymmetric (namely not even). Let us point out that such u^*_ε is the ground state of (2), according to the discussion made in the item (i) above. For example, if $B'' > 0$ and $A'' - C'' < 0$, then the ground state of (2) is asymmetric provided $k < k^* = (C'' - A'')/B''$.

(iii) For semilinear case in higher space dimensions, perturbation type results in Theorem 1.1 have been obtained, see e.g. [4]. The functionals (4) and (5) have natural analogues in higher space dimensions, but they are not smooth any more. Though ground state solutions for (6) have been studied for higher space dimensions in [12, 13] there are difficulties to generalize Theorem 1.1 to higher dimensions, mainly in dealing with estimates involving the quasilinear terms.

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