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EXISTENCE AND MULTIPLICITY RESULTS FOR SOME SUPERLINEAR ELLIPTIC PROBLEMS ON $\mathbb{R}^N$

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Abstract

We study the semilinear elliptic PDE $-\Delta u + b(x)u = f(x,u)$ in $\mathbb{R}^N$. The nonlinearity f will be superlinear and subcritical. We prove the existence of a positive solution under various hypotheses on b. If $b(x) = \lambda a(x) + 1$ and f is odd in u, then we also discuss the dependence of the number of (possibly sign changing) solutions on the parameter λ. We do not assume that b or f have a limit for $|x| \to \infty$.

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1 Introduction

In this paper we study the semilinear elliptic partial differential equation

$$-\Delta u + b(x)u = f(x,u) \qquad x \in \mathbf{R}^N. \tag{1.1}$$

Our interest in this equation results from Rabinowitz' paper [R2] who in turn was motivated by the work of Floer and Weinstein [FW] and of Oh [Oh1-3] on nonlinear Schrödinger equations. They were interested in finding standing wave solutions which leads to the study of (1.1). Similarly, the search for standing (or traveling) waves in nonlinear equations of Klein-Gordon type leads to the study of (1.1). Equation (1.1) appears also in other contexts, for example when one studies reaction-diffusion equations. Then solutions of (1.1) correspond to steady states of the system.

Many papers deal with the autonomous case where b and f are independent of x, or with the radially symmetric case where b and f depend on $|x|$. Then it is natural to look for radially symmetric solutions first. If f grows subcritically then the functional

$$\phi(u) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx - \int_{\mathbf{R}^N} F(x,u)\,dx$$

associated to (1.1) satisfies the Palais-Smale condition if it is restricted to the class of radial functions in $H^1(\mathbf{R}^N)$. Therefore a variety of variational methods can be applied in order to obtain radial solutions. We refer the reader in particular to the work of Strauss [S1], Berestycki and Lions [BL] or Struwe [S2] (and the references cited there) who considered the autonomous version of (1.1). In that case also ODE methods have been applied successfully, for instance by Jones, Küpper and Plakties [JKP] who used a shooting argument. The non-autonomous but radially symmetric case has been studied by Li [L2], for example, who is interested in radial solutions, and in [BW1,2], where both radial and nonradial solutions are found under various growth conditions mainly on f.

If the radial symmetry is lost then, in general, ϕ does not satisfy the Palais-Smale condition any more. This situation has been treated for instance by Ding and Ni [DN], Li [L1] and Rabinowitz [R2], using variational methods like the mountain pass theorem. In these papers the existence of a positive solution is established under various growth conditions on b and f. The main problem in these papers is to circumvent the lack of compactness by either showing that the mountain pass value lies in a range where the Palais-Smale condition holds, using comparison functionals, for instance; or by showing that a weak limit of a Palais-Smale sequence is in fact a non-trivial solution.

In this paper we deal with (1.1) under similar assumptions as in [R2]. We prove that ϕ does indeed satisfy the Palais-Smale condition even under weaker assumptions than those considered in §§1,2 of [R2]. This allows us to prove the existence of infinitely many solutions of (1.1) if f is odd in u. We also treat (1.1) in a situation where the Palais-Smale condition does not hold but where $b(x) = \lambda a(x) + 1$ depends on a parameter λ. Here we investigate the number of solutions of (1.1) as λ increases.

2 Statement of results

As in [R2] we shall be mainly interested in criteria for the existence of multiple solutions of (1.1) depending essentially on the behavior of b at infinity. The nonlinearity f will be superlinear and subcritical. More precisely, throughout the paper we assume the following hypotheses for f.

(f_1) $f \in \mathcal{C}(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ satisfies $f(x,u) = o(|u|)$ as $u \to 0$ uniformly in x.

(f_2) There are constants $a_1, a_2 > 0$ and $s \in (1, (N+2)/(N-2))$ for $N \geq 3$ and $s \in (1, \infty)$ for $N = 1, 2$ such that

$$|f(x,u)| \leq a_1 + a_2|u|^s$$

for every $x \in \mathbf{R}^N$, $u \in \mathbf{R}$.

(f_3) There exists $q > 2$ such that for every $x \in \mathbf{R}^N$, $u \in \mathbf{R} \setminus \{0\}$

$$0 < qF(x,u) = q\int_0^u f(x,v)\,dv \leq uf(x,u).$$

For b we shall first assume the following.

(b_1) $b \in \mathcal{C}(\mathbf{R}^N, \mathbf{R})$ satisfies $b_0 := \inf_{x \in \mathbf{R}^N} b(x) > 0$.

(b_2) For every $M > 0$

$$\mu(\{x \in \mathbf{R}^N : b(x) \leq M\}) < \infty$$

where μ denotes Lebesgue measure in $\mathbf{R}^N$.

Theorem 2.1. *If (b_1), (b_2) and (f_1)-(f_3) hold then there exist a positive and a negative (weak) solution of (1.1). Moreover, if in addition f is odd in u, that is*

(f_4) $f(x,-u) = -f(x,u)$ *for* $x \in \mathbf{R}^N$, $u \in \mathbf{R}$

holds, then (1.1) has infinitely many solutions. These solutions are classical if f is locally Lipschitz continuous.

This result improves Theorem 1.7 of [R2] in two ways. First our hypotheses on both b and f are weaker. In particular, it is not assumed that $b(x) \to \infty$ as $|x| \to \infty$. Secondly, Rabinowitz does not obtain infinitely many solutions if f is odd in u. In [R2] the solutions are obtained as weak limits of certain Palais-Smale sequences of the energy functional

$$u \mapsto \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx - \int_{\mathbf{R}^N} F(x,u)\,dx$$

associated to (1.1). Our main observation is that under the assumptions of Theorem 2.1 the Palais-Smale condition is satisfied. Then the existence of a positive and a negative solution follows from the mountain pass theorem. The weak limits in [R2], 1.7, are in fact strong limits. This is essential in order to prove the existence of infinitely many solutions of (1.1) as in 2.1.

In our next result we weaken hypothesis (b_2) assuming on the other hand that b and f are symmetric with respect to a subgroup of $O(N)$ acting on $x \in \mathbf{R}^N$. It is well known that such symmetries may simplify the problem considerably. For instance, if b and f depend only on $|x|$ then one can look for radial solutions of (1.1). In that case ϕ satisfies the Palais-Smale condition on the space of radial functions. Hypothesis (b_2) is not needed at all. Using a result of P.L. Lions, in [BW1] Bartsch and Willem observed that a weaker symmetry suffices. In the following theorem we allow still weaker symmetries (see also [BCS] in the case of a critical exponent problem).

Theorem 2.2. *Let G be a subgroup of $O(N)$ such that each orbit $Gx = \{gx : g \in G\}$, $x \in \mathbf{R}^N \setminus \{0\}$, has infinitely many elements. Suppose b and f are invariant under G: $b(gx) = b(x)$ and $f(gx,u) = f(x,u)$ for $g \in G$, $x \in \mathbf{R}^N$, $u \in \mathbf{R}$. Then (1.1) has a positive and a negative solution provided (b_1) and (f_1)-(f_3) hold. It has infinitely many solutions if in addition (f_4) holds.*

In fact, in §3 we shall prove a variation of the theorems 2.1, 2.2 where we combine a weakened version of (b_2) with a weakened symmetry condition.

Finally we consider a situation where neither (b_2) nor a symmetry condition is satisfied. The linear term $b(x)u$ depends in an explicit way on a parameter λ and we are interested in the number of solutions of (1.1) as λ increases. More precisely we study

$$-\Delta u + (\lambda a(x) + 1)u = f(x,u) \qquad x \in \mathbf{R}^N. \tag{2.3}$$

Thus we replace $b(x)$ by $b_\lambda(x) = \lambda a(x) + 1$. The following assumptions are required for a.

(a_1) $a \in \mathcal{C}(\mathbf{R}^N, \mathbf{R})$ satisfies $a \geq 0$ and $a^{-1}(0)$ has nonempty interior.

(a_2) There exists $M_0 > 0$ such that

$$\mu(\{x \in \mathbf{R}^N : a(x) \leq M_0\}) < \infty.$$

Theorem 2.4. *Suppose (a_1), (a_2) and (f_1)-(f_3) hold. Then there exists $\lambda_1 > 0$ such that (2.3) has a positive and a negative solution for each $\lambda > \lambda_1$. If in addition (f_4) holds then for each $k \in \mathbf{N}$ there exists $\lambda_k > 0$ such that (2.3) has at least k pairs $\pm u_1, \ldots, \pm u_k$ of nontrivial solutions for $\lambda \geq \lambda_k$.*

Remark 2.5. a) Clearly (a_1) implies (b_1) with $b_0 = \inf_{x \in \mathbf{R}^N} b_\lambda(x)$ normalized to 1. We do not know whether one can get rid of the additional hypothesis that the set $a^{-1}(0) = b_\lambda^{-1}(1)$ has nonempty interior. On the other hand, (a_2) is weaker than (b_2). From (b_2) it follows that for every $M > 0$

$$\lim_{R \to \infty} \mu(\{x \in \mathbf{R}^N : |x| \geq R, b(x) \leq M\}) = 0. \tag{2.6}$$

(a_2) implies that b_λ satisfies (2.6) for some $M_\lambda > 0$. The important point is that $M_\lambda \to \infty$ as $\lambda \to \infty$. Observe that we do not assume that a_λ or f have a limit for $|x| \to \infty$.

b) One can replace $b_\lambda(x) = \lambda a(x) + 1$ by more general functions. Essentially one needs that there is an open set $\Omega \subset \mathbf{R}^N$ with $\{b_\lambda(x) : \lambda \geq 0, x \in \Omega\}$ bounded instead of (a_1). And (a_2) can be replaced by requiring that b_λ satisfies (2.6) for some M_λ with $M_\lambda \to \infty$ as $\lambda \to \infty$.

3 Proof of Theorem 2.1

We shall first prove the existence of a positive solution of (1.1). To do this we replace f by

$$f_+ : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}, \; f_+(x,u) := \begin{cases} f(x,u) & \text{if } u \geq 0\,; \\ 0 & \text{if } u < 0\,. \end{cases}$$

As in [R2] one sees that a nontrivial solution of

$$-\Delta u + b(x)u = f_+(x,u) \qquad x \in \mathbf{R}^n \tag{3.1}$$

must be positive, hence a solution of (1.1). It is well known that solutions of (3.1) are precisely the critical points of the functional

$$\phi(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx - \int_{\mathbf{R}^N} F_+(x,u)\,dx$$

where F_+ is the primitive of f_+. This functional is defined on the space

$$E := \left\{u \in W^{1,2}(\mathbf{R}^N, \mathbf{R}) : \int_{\mathbf{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx < \infty\right\}$$

with the norm

$$\|u\|^2 := \int_{\mathbf{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx.$$

By (b_1) the embedding $E \hookrightarrow W^{1,2}(\mathbf{R}^N, \mathbf{R})$ is continuous. Moreover, because of (f_1), (f_2) we have $\phi \in C^1(E, \mathbf{R})$. We shall apply the mountain pass theorem of [AR] in order to obtain a positive critical value. Clearly, (f_1) implies

$$\psi(u) := \int_{\mathbf{R}^N} F_+(x, u)\, dx = o(\|u\|^2) \quad \text{as } u \to 0,$$

hence, 0 is a local minimum of ϕ. Using (f_3) one sees easily that $\phi(tu) \to -\infty$ as $t \to \infty$ for every $u \in E \setminus \{0\}$. Setting

$$\Gamma := \{\gamma : [0,1] \to E : \ \gamma(0) = 0,\ \gamma(1) \in E \setminus \{0\},\ \phi(t\gamma(1)) \le 0 \text{ for all } t \ge 1\}$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \phi(\gamma(t)),$$

we only have to prove the Palais-Smale condition in order to obtain a critical point of mountain pass type at the level c.

Let $(u_n) \subset E$ be a Palais-Smale sequence, that is, $\phi(u_n)$ is bounded and $\phi'(u_n) \to 0$. We claim that (u_n) has a convergent subsequence. A standard argument yields that (u_n) is bounded (see [R2] for instance). Thus after passing to a subsequence $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ strongly in $L^p_{loc}(\mathbf{R}^N)$ for $2 \le p < 2N/(N-2)$. We first show that $u_n \to u_0$ strongly in $L^2(\mathbf{R}^N)$. For this it suffices to prove that $\alpha_n := \|u_n\|_{L^2} \to \|u_0\|_{L^2}$. Suppose $\alpha_n \to \alpha$ along a subsequence, so $\alpha \ge \|u_0\|_{L^2}$. We claim that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbf{R}^N \setminus B_R} u_n^2(x)\, dx < \varepsilon \qquad \text{uniformly in } n \tag{3.2}$$

where $B_R = B_R(0) = \{x \in \mathbf{R}^N : |x| \le R\}$. If (3.2) holds then $u_n \to u_0$ strongly in $L^2(\mathbf{R}^N)$ because $u_n|_{B_R} \to u_0|_{B_R}$ in $L^2(B_R)$, hence:

$$\begin{aligned} \|u_0\|_{L^2(\mathbf{R}^N)} &= \|u_0\|_{L^2(B_R)} + \|u_0\|_{L^2(\mathbf{R}^N \setminus B_R)} \\ &\ge \lim_{n\to\infty} \|u_n\|_{L^2(B_R)} \\ &\ge \lim_{n\to\infty} \|u_n\|_{L^2(\mathbf{R}^N)} - \lim_{n\to\infty} \|u_n\|_{L^2(\mathbf{R}^N \setminus B_R)} \\ &\ge \alpha - \varepsilon. \end{aligned}$$

It remains to prove (3.2). We fix $\varepsilon > 0$ and choose constants $M > \frac{2}{\varepsilon} \sup_n \|u_n\|^2$, $p \in (1, N/(N-2))$ and

$$C \ge \sup_{u \in E \setminus \{0\}} \frac{\|u_n\|^2_{L^{2p}}}{\|u_n\|^2} \tag{3.3}$$

Let p' satisfy $1/p + 1/p' = 1$. Now (b_2) implies that for $R > 0$ large enough

$$\mu(\{x \in \mathbf{R}^N \setminus B_R : b(x) < M\}) \le \left(\frac{\varepsilon}{2C \cdot \sup_n \|u_n\|^2}\right)^{p'}. \tag{3.4}$$

We set

$$A := \{x \in \mathbf{R}^N \setminus B_R : b(x) \geq M\}$$

and

$$B := \{x \in \mathbf{R}^N \setminus B_R : b(x) < M\}.$$

Then by our choice of M

$$\int_A u_n^2 dx \leq \int_{\mathbf{R}^N} \frac{b(x)}{M} u_n^2 \, dx \leq \|u_n\|^2/M \leq \varepsilon/2$$

Moreover, the Hölder inequality, (3.3) and (3.4) imply

$$\begin{aligned} \int_B u_n^2 \, dx &\leq \left(\int_B |u_n|^{2p} dx\right)^{1/p} \cdot \left(\int_B 1 \, dx\right)^{1/p'} \\ &= \|u_n\|_{L^{2p}}^2 \cdot \mu(B)^{1/p'} \\ &\leq C\|u_n\|^2 \cdot \mu(B)^{1/p'} \\ &\leq \varepsilon/2. \end{aligned}$$

Therefore we obtain

$$\int_{\mathbf{R}^N \setminus B_R} u_n^2 \, dx = \int_A u_n^2 \, dx + \int_B u_n^2 \, dx \leq \varepsilon .$$

Thus we have proved that $u_n \to u_0$ in $L^2(\mathbf{R}^N)$. Now one can either use the Gagliardo-Nirenberg inequality or a result of P.L. Lions (see Lemma 4.3 below) in order to see that $u_n \to u_0$ in $L^p(\mathbf{R}^N)$ for $2 \leq p < 2N/(N-2)$. With $s \in (1, (N+2)/(N-2))$ (respectively $s \in (1, \infty)$ for $N = 1, 2$) from assumption (f_2) and $p := s + 1$, one observes that the functional ψ belongs to $C^1(L^p(\mathbf{R}^N), \mathbf{R})$. Let $i : E \hookrightarrow L^p(\mathbf{R}^N)$ denote the inclusion and $D : E \to E^*$ the duality map. Then we know that $i(u_n) \to i(u_0)$ in $L^p(\mathbf{R}^N)$ and

$$u_n = D^{-1} \circ \phi'(u_n) + D^{-1} \circ \psi' \circ i(u_n) \to 0 + D^{-1} \circ \psi' \circ i(u_0)$$

strongly in E. This proves the Palais-Smale condition, hence the existence of a positive solution u_1 of (1.1) of mountain pass type.

The existence of a negative solution u_2 of mountain pass type follows analogously replacing f_+ by

$$f_- : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}, \; f_-(x,u) := \begin{cases} 0 & \text{if } u \geq 0; \\ f(x,u) & \text{if } u < 0. \end{cases}$$

The existence of infinitely many solutions, in fact of an unbounded sequence of critical points of ϕ if f is odd (so ϕ is even) is a consequence of the symmetric mountain pass theorem (see [R1]) applied to the functional ϕ with f_+ replaced by f.

Remark 3.5. From the proof of Theorem 2.1, we can see in fact we have proved that the imbedding from E to $L^2(\mathbf{R}^N)$ is compact. By the Gagliardo-Nirenberg inequality the imbedding from E to $L^p(\mathbf{R}^N)$ is also compact for $2 \leq p < \frac{2N}{N-2}$. Under the coercivity condition, that is, $b(x) \to \infty$ as $|x| \to \infty$, similar compact imbedding results are due to Omana and Willem [OW] in the context of Hamiltonian systems (i.e. $N = 1$) and Costa [C] in the context of elliptic systems. Clearly the coercivity condition implies (b_2). With the aid of this, infinitely many homoclinic solutions are obtained in [OW] if the system is odd. Theorem 2.1 can be considered as an improvement and generalization of the results in [OW] and [C]. We thank M. Willem and the referee for pointing out these references of which we were not aware.

4 Proof of Theorem 2.2

In this section we shall prove a result which generalizes the theorems 2.1 and 2.2. In order to do this we combine a symmetry condition on b and f with a version of (b_2). More precisely, we assume the following symmetry condition.

(S) b and f are invariant under the action of a closed subgroup G of $O(N)$, that is, $b(gx) = b(x)$ and $f(gx, u) = f(x, u)$. Moreover, $\mathbf{R}^N$ splits as $\mathbf{R}^N \cong V \oplus W$ with G-invariant subspaces V and $W = V^\perp$ such that for each $v \in V \setminus \{0\}$ the orbit $Gv = \{gv : g \in G\}$ has infinitely many elements.

Observe that we do not assume anything on the action of G on W. In particular, this action may be trivial. If (S) holds we can weaken (b_2) as follows.

(b_3) For every $M > 0$ and every $R > 0$

$$\mu(\{x = v + w \in \mathbf{R}^N = V \oplus W : b(x) \leq M,\ |v| \leq R\}) < \infty.$$

Theorem 4.1. *If (S), (b_1), (b_3) and (f_1)–(f_3) hold then there exists a positive and a negative solution of (1.1). These solutions are G-invariant.*

Theorem 2.1 corresponds to the case $V = 0$, Theorem 2.2 to the case $W = 0$.

Proof. We use the same notation as in §3. Observe that G acts on E via

$$(gu)(x) := u(g^{-1}x) \quad \text{for } g \in G,\ u \in E,\ x \in \mathbf{R}^N.$$

This action preserves the norm because of (S):

$$\|gu\|^2 = \int_{\mathbf{R}^N} (|\nabla(gu)(x)|^2 + b(x)(gu)^2(x))\,dx$$

$$\begin{aligned} &= \int_{\mathbf{R}^N} (|g(\nabla u)(g^{-1}x)|^2 + b(g^{-1}x)u^2(g^{-1}x))\,dx \\ &= \int_{\mathbf{R}^N} (|\nabla(u)(y)|^2 + b(y)u^2(y))\,dy \\ &= \|u\|^2 \end{aligned}$$

Also ϕ is G-invariant because f_+ and its primitive F_+ are G-invariant, hence $D^{-1}\phi' : E \to E$ is equivariant: $D^{-1}\phi'(gu) = gD^{-1}\phi'(u)$. This implies $D^{-1}\phi'(E^G) \subset E^G$ and $D^{-1}\phi'(E^{G\perp}) \subset E^{G\perp}$. Here $E^G := \{u \in E : \ gu = u$ for all $g \in G\}$ consists of the G-invariant functions u in E. Thus it suffices to find critical points of the restriction $\phi_0 := \phi|E^G$. We shall show that ϕ_0 satisfies the Palais-Smale condition. The theorem follows then as in §3.

Let $(u_n) \subset E^G$ be a Palais-Smale sequence. Then as in §3 it must be bounded and after passing to a subsequence $u_n \rightharpoonup u_0$ in E^G and $u_n \to u_0$ in $L^p_{loc}(\mathbf{R}^N)$ for $2 \le p < 2N/(N-2)$. Fixing any $r > 0$, we claim that

$$\sup_{x\in\mathbf{R}^N} \int_{B_r(x)} |u_n - u_0|^2\,dx \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Postponing the proof of (4.2) we show how (PS) follows. The following lemma is due to P.L. Lions ([L3], Lemma I.1).

Lemma 4.3. *If (v_n) is bounded in $W^{1,2}(\mathbf{R}^N)$ and if*

$$\sup_{x\in\mathbf{R}^N} \int_{B_r(x)} |v_n|^2\,dx \to 0 \quad \text{as } n \to \infty$$

then $v_n \to 0$ in $L^p(\mathbf{R}^N)$ for any $2 \le p < 2N/(N-2)$.

Thus (4.2) implies $u_n \to u_0$ in $L^p(\mathbf{R}^N)$ with $p = s+1$. Then $u_n \to u_0$ in E follows as in §3.

It remains to prove (4.2). For $R > r > 0$ we define

$$\begin{aligned} m(R,r) := \min_{|v|=R, v\in V} \sup\{n \in \mathbf{N} :\ &\text{there exists } g_1, \dots, g_n \in G \\ &\text{such that } B_r(g_iv) \cap B_r(g_jv) = \emptyset \text{ for all } i \ne j\} \end{aligned}$$

We first show that

$$\lim_{R\to\infty} m(R,r) = \infty. \tag{4.4}$$

Clearly, $m(R,r) = m(\varepsilon R/r, \varepsilon)$, so (4.4) follows from

$$\lim_{\varepsilon\to 0} m(1,\varepsilon) = \infty. \tag{4.5}$$

In order to prove (4.5) suppose to the contrary that there exist $k \in \mathbf{N}$ and a sequence $v_n \in V$ with $|v_n| = 1$ such that for each $g_1, \dots, g_k \in G$ there are $i \ne j$ with $B_{1/n}(g_iv_n) \cap B_{1/n}(g_jv_n) \ne \emptyset$. For an accumulation point v of (v_n) we would then have $|Gv| < k$ because otherwise there are $g_1, \dots, g_k \in G$ and $\varepsilon > 0$ with $B_\varepsilon(g_i v) \cap B_\varepsilon(g_j v) = \emptyset$ if $i \neq j$. Thus if n is large enough we have $1/n < \varepsilon/2$ and $v_n \in B_{\varepsilon/2}(v)$, hence $B_{1/n}(g_i v_n) \cap B_{1/n}(g_j v_n) = \emptyset$, a contradiction. Now (4.5) and (4.4) follow since $|Gv| = \infty$ by (S).

Using (4.4) there exists $R_1 > 0$ such that for $x = v + w$ with $|v| \geq R_1$

$$\int_{B_r(x)} |u_n - u_0|^2 \, dx \leq \|u_n\|_{L^2}^2 / m(R_1, r) \leq \varepsilon$$

uniformly in n. Using (b_3) one can show as in the proof of (3.2) that for $R_2 > 0$ big enough and $x = v + w$ with $|v| \leq R_1$, $|w| \geq R_2$

$$\int_{B_r(x)} |u_n - u_0|^2 \, dx \leq \varepsilon$$

uniformly in n. Using $u_n \to u_0$ in $L^2_{loc}(\mathbf{R}^N)$ we obtain (4.2) immediately. This proves Theorem 4.1. □

Condition (4.4) is a variation of a condition introduced by Willem [W].

Remark 4.6. Our argument also shows that the imbedding from E^G to $L^2(\mathbf{R}^N)$ is compact.

5 Proof of Theorem 2.4

We consider the functional

$$\phi_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla u|^2 + (\lambda a(x) + 1) u^2) \, dx - \int_{\mathbf{R}^N} F(x, u) \, dx$$

for $u \in E := \{u \in W^{1,2}(\mathbf{R}^N, \mathbf{R}) : \|u\|^2 < \infty\}$ where

$$\|u\|^2 := \int_{\mathbf{R}^N} (|\nabla u|^2 + (a(x) + 1) u^2) \, dx.$$

Then $\phi_\lambda \in C^1(E, \mathbf{R})$ for every $\lambda \geq 0$. By (f_1) ϕ_λ has a local minimum at 0 and by (f_3) $\phi_\lambda(tu) \to -\infty$ as $t \to \infty$ for $u \in E \setminus 0$. As in §3 we set

$$\Gamma_\lambda := \{\gamma : [0,1] \to E : \gamma(0) = 0, \gamma(1) \in E \setminus \{0\}, \phi_\lambda(t\gamma(1)) \leq 0 \text{ for all } t \geq 1\}$$

and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{0 \leq t \leq 1} \phi_\lambda(\gamma(t)).$$

By Ekeland's variational principle or the quantitative deformation lemma (see [W]) there exist sequences $(u_n^\lambda)_n$ such that $\phi_\lambda(u_n^\lambda) \to c_\lambda$ and $\phi'_\lambda(u_n^\lambda) \to 0$ as $n \to \infty$. It follows easily that $(u_n^\lambda)_n$ is bounded in E, hence $u_n^\lambda \rightharpoonup u_0^\lambda$ in E and $u_n^\lambda \to u_0^\lambda$ in $L^p_{loc}(\mathbf{R}^N)$ for $2 \leq p < 2N/(N-2)$. And one may check that u_0^λ is a (weak) solution of (2.3). The first part of Theorem 2.4 follows if we can show that $u_0^\lambda \neq 0$ for λ big enough. This is a consequence of the following two lemmas. We write $u|_A$ for the restriction of $u \in E$ to $A \subset \mathbf{R}^N$ and $\|u\|_{s+1}$ for the L^{s+1}-norm.

Lemma 5.1. *There exists $\alpha > 0$ such that*

$$\liminf_{n\to\infty} \|u_n^\lambda\|_{s+1}^{s+1} \geq \alpha \quad \textit{for every } \lambda \geq 0.$$

Lemma 5.2. *For every $\beta > 0$ there exist $\lambda_1 > 0$ and $R > 0$ such that for $\lambda \geq \lambda_1$*

$$\|u_n^\lambda|_{\mathbf{R}^N\setminus B_R}\|_{s+1}^{s+1} < \beta\,.$$

From these lemmas we obtain with $\beta := \alpha/2$ and λ_1, R as in 5.2

$$\|u_0^\lambda|_{B_R}\|_{s+1}^{s+1} = \lim_{n\to\infty} \|u_n^\lambda|_{B_R}\|_{s+1}^{s+1} \geq \alpha/2 > 0$$

provided $\lambda \geq \lambda_1$. Thus $u_0^\lambda \neq 0$ for $\lambda \geq \lambda_1$. It remains to prove the two lemmas. For simplicity we drop the superscript λ in the sequel and write u_n for u_n^λ.

Proof of Lemma 5.1. Since

$$\phi_\lambda(u) \geq \phi_0(u) = \frac{1}{2}\|u\|_{W^{1,2}}^2 - o(\|u\|_{W^{1,2}}^2) \quad \text{for } u \to 0$$

there exists $\underline{c} > 0$ such that $c_\lambda \geq \underline{c}$ for every $\lambda \geq 0$. Next we observe that

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty} (\phi_\lambda(u_n) - \frac{1}{q}\phi_\lambda'(u_n)u_n) \\ &\geq \limsup_{n\to\infty} \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbf{R}^N} (|\nabla u_n|^2 + b_\lambda(x)u_n^2)\,dx \end{aligned} \tag{5.3}$$

and consequently (for $\lambda \geq 1$)

$$\|u_n\|^2 \leq \frac{2q}{q-2}c_\lambda + o(1) \quad \text{as } n \to \infty. \tag{5.4}$$

Now (f_1) and (f_2) imply that for every $\varepsilon > 0$ there exists A_ε such that

$$\frac{1}{2}f(x,u)u - F(x,u) \leq \varepsilon u^2 + A_\varepsilon |u|^{s+1}$$

for every $x \in \mathbf{R}^N$, $u \in \mathbf{R}$. This yields, using (5.4)

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty} (\phi_\lambda(u_n) - \frac{1}{2}\phi_\lambda'(u_n)u_n) \\ &= \lim_{n\to\infty} \int_{\mathbf{R}^N} \left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\leq \liminf_{n\to\infty} \int_{\mathbf{R}^N} (\varepsilon|u_n|^2 + A_\varepsilon|u_n|^{s+1})\,dx \\ &\leq \liminf_{n\to\infty} \left(\frac{2q\varepsilon}{q-2}c_\lambda + A_\varepsilon\|u_n\|_{s+1}^{s+1}\right) \end{aligned}$$

Setting $\varepsilon_0 := (q-2)/4q$ we have

$$\liminf_{n\to\infty} \|u_n\|_{s+1}^{s+1} \geq c_\lambda/2A_{\varepsilon_0} \geq \underline{c}/2A_{\varepsilon_0} =: \alpha\,.$$

□

Proof of Lemma 5.2. Let $v \in E \setminus \{0\}$ have support in $a^{-1}(0)$. This exists because of (a_1). Then by definition of c_λ:

$$\begin{aligned} c_\lambda &\leq \max_{t\geq 0} \phi_\lambda(tv) \\ &= \max_{t\geq 0} \left(\frac{t^2}{2} \int_{\mathbf{R}^N} (|\nabla v|^2 + v^2)\, dx - \int_{\mathbf{R}^N} F(x, tv)\, dx \right) \\ &=: \bar{c}. \end{aligned}$$

Next, for $R > 0$ we set

$$A(R) := \{x \in \mathbf{R}^N : |x| > R,\ a(x) \geq M_0\}$$

and

$$B(R) := \{x \in \mathbf{R}^N : |x| > R,\ a(x) < M_0\}.$$

Then we have, using (5.3)

$$\begin{aligned} \int_{A(R)} u_n^2\, dx &\leq \frac{1}{\lambda M_0 + 1} \int_{\mathbf{R}_N} (\lambda a(x) + 1) u_n^2\, dx \\ &\leq \frac{1}{\lambda M_0 + 1} \int_{\mathbf{R}_N} (|\nabla u_n|^2 + (\lambda a(x) + 1) u_n^2)\, dx \\ &\leq \frac{1}{\lambda M_0 + 1} \left(\frac{2q}{q-2} c_\lambda + o(1) \right) \\ &\leq \frac{1}{\lambda M_0 + 1} \left(\frac{2q}{q-2} \bar{c} + o(1) \right) \end{aligned} \tag{5.5}$$

as $n \to \infty$. Moreover, using the Hölder inequality and (5.4) we obtain for $p \in (1, N/(N-2))$

$$\begin{aligned} \int_{B(R)} u_n^2\, dx &\leq \left(\int_{\mathbf{R}^N} |u_n|^{2p}\, dx \right)^{1/p} \cdot \left(\int_{B(R)} 1\, dx \right)^{1/p'} \\ &\leq C_1 \|u_n\|^2 \cdot \mu(B(R))^{1/p'} \\ &\leq C_1 \frac{2q}{q-2} c_\lambda \cdot \mu(B(R))^{1/p'} + o(1) \\ &\leq C_1 \frac{2q}{q-2} \bar{c} \cdot \mu(B(R))^{1/p'} + o(1) \end{aligned} \tag{5.6}$$

as $n \to \infty$. Here $C_1 = C_1(N,p)$ is a positive constant. Setting $\theta := N(s-1)/2(s+1)$, the Gagliardo-Nirenberg inequality yields ($C_2 = C_2(N,s)$ is a positive constant):

$$\begin{aligned}\int_{\mathbf{R}^N\setminus B_R} |u_n|^{s+1}dx &\leq C_2\|\nabla u_n|_{\mathbf{R}^N\setminus B_R}\|_2^{\theta(s+1)} \cdot \|u_n|_{\mathbf{R}^N\setminus B_R}\|_2^{(1-\theta)(s+1)} \\ &\leq C_2\|u_n\|^{\theta(s+1)} \cdot \left(\int_{A(R)} u_n^2\,dx + \int_{B(R)} u_n^2\,dx\right)^{(1-\theta)(s+1)/2}\end{aligned}$$

Now observe that $\|u_n\|$ is bounded independent of n and λ because of (5.4) and $c_\lambda \leq \bar{c}$. By (5.5) the first summand on the right can be made arbitrarily small if λ is large. Using (5.6) the second summand will be arbitrarily small if R is large because $\mu(B(R)) \to 0$ as $R \to \infty$ by (a_2). This proves Lemma 5.2. □

Now suppose that f is odd in the u-variable, so $\phi_\lambda \in \mathcal{C}^1(E;\mathbf{R})$ is even. Let $\Omega \subset \operatorname{int} a^{-1}(0)$ be an open bounded domain with smooth boundary and consider $F := W_0^{1,2}(\Omega)$ as a subspace of $E \subset W^{1,2}(\mathbf{R}^N)$. Observe that $\phi_\lambda(u) = \phi_0(u)$ is independent of λ for $u \in F$. We look at the eigenvalue problems

$$-\Delta u + (a(x)+1)u = \mu u \qquad x \in \mathbf{R}^N,$$

with eigenfunctions $e_j \in E$, $j \in \mathbf{N}$, and

$$\begin{aligned}-\Delta u + u &= \mu u \qquad x \in \Omega \\ u|\partial\Omega &= 0\end{aligned}$$

with eigenfunctions $f_j \in F \subset E$, $j \in \mathbf{N}$. We set $E_k := \operatorname{span}\{e_1,\dots,e_k\}$ and $F_k := \operatorname{span}\{f_1,\dots,f_k\}$. Since $\phi_\lambda(tu) \to -\infty$ as $t \to \infty$ for every $u \in E\setminus\{0\}$ we see that ϕ_λ satisfies the following property:

(ϕ_1) For each $k \geq 1$ there exists $R_k > 0$ with $\phi_\lambda(u) \leq 0$ for $u \in F_k$, $\|u\| \geq R_k$.

In fact, R_k is independent of λ because $\phi_\lambda|F = \phi_0|F$. Moreover, using (f_2) a standard argument yields

(ϕ_2) $\underline{c}_k := \sup_{\rho>0} \inf_{u\in E_{k-1}^\perp, \|u\|=\rho} \phi_0(u) \to \infty$ as $k \to \infty$.

Now we define for $k \in \mathbf{N}$ and $\lambda \geq 0$

$$B_k := \{u \in F_k : \|u\| \leq R_k\},$$

$$\Gamma_k := \{\gamma\colon B_k \to E:\ \gamma \text{ is odd, continuous and } \gamma(u) = u \text{ if } \|u\| = R_k\}$$

and

$$c_k^\lambda := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \phi_\lambda(\gamma(u)).$$

These are almost critical values, that is, there exist sequences $(u_{k,n}^\lambda)_n$ with $\phi_\lambda(u_{k,n}^\lambda) \to c_k^\lambda$ and $\phi_\lambda'(u_{k,n}^\lambda) \to 0$ as $n \to \infty$. From this it follows that $(u_{k,n}^\lambda)_n$ is bounded in E, so $u_{k,n}^\lambda \rightharpoonup u_k^\lambda$ in E and $u_{k,n}^\lambda \to u_k^\lambda$ in $L_{loc}^p(\mathbf{R}^N)$, $2 \leq p < 2N/(N-2)$.

Lemma 5.7. $\underline{c}_k \le c_k^\lambda \le \bar{c}_k := \max_{u\in B_k} \phi_0(u)$ *for every* $\lambda \ge 0$, $k \in \mathbb{N}$.

Proof. The inequality $c_k^\lambda \le \bar{c}_k$ is trivial since $\phi_\lambda|F = \phi_0|F$. In order to see $c_k^\lambda \ge \underline{c}_k$ we show that for every $\gamma \in \Gamma_k$ and every $\rho \in (0, R_k)$ there exists $u \in B_k$ with $\gamma(u) \in E_{k-1}^\perp$ and $\|\gamma(u)\| = \rho$. Then $c_k^\lambda \ge \underline{c}_k$ follows from the definition of R_k and $\underline{c}_k$ in (ϕ_1), (ϕ_2) together with the fact $\phi_\lambda \ge \phi_0$ for $\lambda \ge 0$.

Given γ and ρ as above we set

$$\mathcal{O} := \{u \in B_k : \|\gamma(u)\| < \rho\}$$

which is an open neighborhood of 0 in F_k and satisfies $\overline{\mathcal{O}} \subset \operatorname{int} B_k$ because $\gamma(u) = u$ for $u \in \partial B_k$. Let $p_{k-1} : E \to E_{k-1}$ denote the orthogonal projection and set

$$h := p_{k-1} \circ \gamma|\overline{\mathcal{O}} : \overline{\mathcal{O}} \to E_{k-1}$$

This is an odd continuous map. By Borsuk's theorem there exists $u \in \partial\mathcal{O}$ with $h(u) = 0$. Clearly this implies $\gamma(u) \in E_{k-1}^\perp$ and $\|\gamma(u)\| = \rho$. □

Lemma 5.8. *There exists a sequence* $\alpha_k \to \infty$ *such that*

$$\liminf_{n\to\infty} \|u_{k,n}^\lambda\|_{s+1}^{s+1} \ge \alpha_k \qquad \text{for every } \lambda \ge 0.$$

Proof. The proof of Lemma 5.1 applies with α replaced by $\alpha_k := \underline{c}_k/2A_{\varepsilon_0}$. Moreover, $\alpha_k \to \infty$ according to (ϕ_2). □

Lemma 5.9. *There exists* $\beta_k > 0$ *such that*

$$\limsup_{n\to\infty} \|u_{k,n}^\lambda\|_{s+1}^{s+1} \le \beta_k \qquad \text{for every } \lambda \ge 1.$$

Proof. As in the proof of (5.4) we see that for $\lambda \ge 1$

$$\|u_{k,n}^\lambda\|^2 \le \frac{2q}{q-2}\, c_k^\lambda + o(1) \qquad \text{as } n \to \infty.$$

Using the Sobolev inequality and Lemma 5.7 the result follows. □

Next we choose a sequence $k_i \to \infty$ such that $\alpha_{k_{i+1}} > 4\beta_{k_i}$ for each $i \in \mathbb{N}$. For simplicity of notation we set $\gamma_i := \alpha_{k_i}$, $\delta_i := \beta_{k_i}$, $v_{i,n}^\lambda := u_{k_i,n}^\lambda$, and $v_i^\lambda := u_{k_i}^\lambda$ for the weak limit of the $v_{i,n}^\lambda$.

Lemma 5.10. *For each* $m \in \mathbb{N}$ *there exists* $\lambda_m > 0$ *such that*

$$\|v_i^\lambda\|_{s+1}^{s+1} \ge \gamma_i/2 \qquad \text{for } \lambda \ge \lambda_m,\ i = 1, \ldots, m.$$

Postponing the proof of Lemma 5.10 we first show how the existence of m nontrivial solutions $\pm v_i^\lambda$, $i = 1, \ldots, m$ follows if $\lambda \ge \lambda_m$. Namely, applying 5.10 and 5.9 we obtain for $\lambda \ge \lambda_m$, $i = 1, \ldots, m$

$$\gamma_i/2 \le \|v_i^\lambda\|_{s+1}^{s+1} \le \delta_i \le \gamma_{i+1}/4 < \gamma_{i+1}/2 \le \|v_{i+1}^\lambda\|_{s+1}^{s+1}.$$

Thus the v_i^λ, $i = 1, \ldots, m$ are different from 0 and have increasing L^{s+1}-norms, provided $\lambda \geq \lambda_m$.

It remains to prove Lemma 5.10. As in the proof of (5.5) and (5.6) we see that for $R > 0$

$$\begin{aligned} \int_{\mathbf{R}^N \setminus B_R} |u_{k,n}^\lambda|^2 \, dx &= \int_{A(R)} |u_{k,n}^\lambda|^2 \, dx + \int_{B(R)} |u_{k,n}^\lambda|^2 \, dx \\ &\leq \frac{2q}{q-2} \bar{c}_k \cdot \left(\frac{1}{\lambda M_0 + 1} + C_1 \mu(B(R))^{1/p'} \right) + o(1) \end{aligned}$$

as $n \to \infty$. Again $p \in (1, N/(N-2))$ and $C_1 = C_1(N, p)$ is a positive constant. If we fix $m \in \mathbf{N}$ then the right hand side can be made arbitrarily small for all $k \leq k_m$ provided λ and R are large enough. By the Gagliardo-Nirenberg inequality we obtain as in the proof of Lemma 5.2

$$\int_{\mathbf{R}^N \setminus B_R} |u_{k,n}^\lambda|^{s+1} dx \leq C_2 \|u_{k,n}^\lambda\|^{\theta(s+1)} \cdot \|u_{k,n}^\lambda|_{\mathbf{R}^N \setminus B_R}\|_2^{(1-\theta)(s+1)}.$$

Since $\|u_{k,n}^\lambda\|$ is bounded independent of $k \leq k_m$, $n \in \mathbf{N}$ and $\lambda > 0$ there exists $\lambda_m > 0$ and $R_m > 0$ such that

$$\int_{\mathbf{R}^N \setminus B_R} |u_{k,n}^\lambda|^{s+1} \, dx \leq \alpha_k / 2$$

for all $k \leq k_m$, $n \in \mathbf{N}$, $\lambda \geq \lambda_m$, $R \geq R_m$. This implies for $k = k_1, \ldots, k_m$, $\lambda \geq \lambda_m$, $R \geq R_m$

$$\begin{aligned} \|v_i^\lambda|_{B_R}\|_{s+1}^{s+1} &= \lim_{n\to\infty} \|v_{i,n}^\lambda|_{B_R}\|_{s+1}^{s+1} \\ &\geq \liminf_{n\to\infty} \|v_{i,n}^\lambda\|_{s+1}^{s+1} - \limsup_{n\to\infty} \|v_{i,n}^\lambda|_{\mathbf{R}^N \setminus B_R}\|_{s+1}^{s+1} \\ &\geq \gamma_i - \gamma_i/2 = \gamma_i/2. \end{aligned}$$

This proves Lemma 5.10. □

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