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Nodal type bound states of Schrödinger equations via invariant set and minimax methods

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Abstract

For nonlinear Schrödinger equations in the entire space we present new results on invariant sets of the gradient flows of the corresponding variational functionals. The structure of the invariant sets will be built into minimax procedures to construct nodal type bound state solutions of nonlinear Schrödinger type equations.

Keywords: Invariant sets; Bound state nodal solutions; Nonlinear Schrödinger equations; Minimax method

1. Introduction

The principal project of this paper is to investigate the structure of invariant sets of the associated gradient flows for nonlinear Schrödinger equations in the entire space, and in conjunction with minimax method to construct nodal type bound state solutions. In particular, we shall study how the structure of global invariant sets depends upon the local behavior of the flow near the trivial critical point 0. The novelty of our work is to discover a new family of invariant sets when there is a hyperbolic structure near the trivial critical point 0. As applications we shall provide multiplicity results of nodal type bound state solutions for nonlinear Schrödinger type equations. More precisely,

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as a model problem we consider the existence of nodal (sign-changing) solutions for nonlinear time-independent Schrödinger equations of the form

$$-\Delta u + V_\lambda(x)u = f(x,u) \quad \text{in } \mathbf{R}^N, \tag{1.1}$$

which satisfy $u(x) \to 0$ as $|x| \to \infty$. This type of equations arise also from study of standing wave solutions of time-dependent nonlinear Schrödinger equations. The potential function $V_\lambda(x) := \lambda g(x) + 1$ satisfies the following conditions:

(V_1) $g \in C(\mathbf{R}^N, \mathbf{R})$ satisfies $g \geqslant 0$ and $\Omega := \mathrm{int}(g^{-1}(0))$ is nonempty.
(V_2) There exist $M_0 > 0$ and $r_0 > 0$ such that

$$\lim_{|y|\to\infty} m\Big(\{x \in \mathbf{R}^N : |x-y| \leqslant r_0\} \cap \{x \in \mathbf{R}^N : g(x) \leqslant M_0\}\Big) = 0,$$

where m denotes the Lebesgue measure on $\mathbf{R}^N$.
(V_3) $\overline{\Omega} = g^{-1}(0)$ and $\partial\Omega$ is locally Lipschitz.

As $\lambda \to \infty$, V_λ has a steep potential well, and we are interested in finding solutions trapped in the potential well. Under the above conditions, the linear operator $-\Delta + V_\lambda$ may have a finite number of eigenvalues below the infimum of the essential spectrum [3]. Obviously, these eigenvalues (bound states to the linear problem), except the first one, have nodal eigenfunctions. We shall show that under suitable asymptotically linear perturbations $f(x,u)$, the nonlinear problem (1.1) has multiple bound state nodal solutions resembling the nodal structure of the linear problem.

In order to state our conditions on f we introduce the following eigenvalue problem on Ω (cf. [25, Proposition A.1]):

$$\left.\begin{aligned} -\Delta u + u &= \nu u \quad \text{in } \Omega \\ u &= 0 \quad\ \ \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

Let the eigenvalues of this problem be denoted by $0 < \nu_1 < \nu_2 < \nu_3 < \cdots$, which will occasionally be written as $0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$, counting their multiplicity. Using the convention $\nu_0 := -\infty$, we make the following assumptions on f:

(f_0) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$.
(f_1) $f(x,s)$ is odd in s.
(f_2) There exists an $m \in \mathbf{N}_0 := \{0,1,2,\ldots\}$ such that uniformly in $x \in \mathbf{R}^N$,

$$\nu_m < \liminf_{|s|\to 0} \frac{f(x,s)}{s} \leqslant \limsup_{|s|\to 0} \frac{f(x,s)}{s} < \nu_{m+1}.$$

(f_3) There exists an $n \in \mathbf{N}_0$, $n \neq m$ such that uniformly in $x \in \mathbf{R}^N$,

$$\nu_n < \liminf_{|s|\to\infty} \frac{f(x,s)}{s} \leqslant \limsup_{|s|\to\infty} \frac{f(x,s)}{s} < \nu_{n+1}.$$

(f_4) There exists a $c_0 > 0$ such that $|f(x,s)| \leqslant c_0|s|$ for all (x,s).
(f_5) There exists an $L \geqslant 0$ such that $f(x,s) + Ls$ is increasing in s.

Let $\dim(v_i)$ denote the dimension of the eigenspace associated with the eigenvalue v_i. Set

$$d_k := \sum_{i=1}^{k} \dim(v_i) \quad \text{and} \quad d_0 := 0.$$

Theorem 1.1. *Assume* (V_1)–(V_3) *and* (f_0)–(f_5). *Then there exists a* $\Lambda > 0$ *such that for all* $\lambda \geqslant \Lambda$, *Eq.* (1.1) *has at least* $|d_m - d_n|$ *pairs of nodal solutions provided* $\min\{m,n\} \geqslant 1$, *and at least* $|d_m - d_n| - 1$ *pairs of nodal solutions if* $\min\{m,n\} = 0$.

Remark 1.2. In the case $\min\{m,n\} = 0$, there is also a pair of signed solutions $u > 0$, $-u < 0$.

Remark 1.3. Though the existence result is stated and will be proved in detail only for the model Eq. (1.1) with the nonlinearity given above, the approach we shall take will be useful for more general types of nonlinearities. The understanding of invariant sets for gradient flows has been the center issue in dealing with nodal solutions. The key new ingredient we provide here is the construction and the structure of a new family of invariant sets of the associated gradient flows (this is done in Section 4), which, to our knowledge, is the first nontrivial construction of invariant sets near a saddle critical point. By using a combination of invariant sets method and minimax method as is done in this paper, many multiplicity results on nodal solutions for bounded domains like in [1,4,14] can be generalized to nonlinear Schrödinger equations in the entire space, for example, superlinear problems with a saddle point at 0, asymptotically linear problems with resonance, nonlinear eigenvalue problems, etc. We leave the precise statements to interested readers.

We finish the section with some historical comments on related work and methods involved, and outlining in more detail our approach.

In the case where f is assumed to be superlinear, nonlinear Schrödinger type equations have received a lot of attention in the past. It is only recently that nonlinear Schrödinger type equations with asymptotically linear terms have been studied. In [22,23] Stuart and Zhou studied radially symmetric problem. More general situations were considered in [8,11,12,15,26]. In most of these papers, the potential function is either periodic or autonomous at infinity. Asymptotically linear problems with potentials in this paper have been studied in [24,25], in which multiple solutions were constructed without giving nodal information about the solutions. On the other hand, results like Theorem 1.1 for problems in bounded domains have been given in [14].

In this paper, we shall construct nodal solutions for Eq. (1.1) by building upon the general idea of combining invariant sets with minimax method (which has been very successful for bounded domain problems, e.g., [1,4,14,18]) and by developing new techniques which will overcome difficulties for unbounded domain problems. To describe our basic idea and approach, let us introduce some notations. It is well known that weak solutions of (1.1) correspond to critical points of

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + V_\lambda u^2)\,dx - \int_{\mathbf{R}^N} F(x,u)\,dx,$$

in $H_\lambda := \{u \in H^1(\mathbf{R}^N) : \|u\|_\lambda < \infty\}$. Here $F(x,u) := \int_0^u f(x,s)\,ds$ and $\|\cdot\|_\lambda$ is the norm induced by the inner product

$$(u,\ v)_\lambda := \int_{\mathbf{R}^N} (\nabla u \cdot \nabla v + V_\lambda uv)\,dx.$$

The main idea of this paper is to construct certain invariant sets of the gradient flow associated to the energy functional I_λ so that all positive and negative solutions are contained in these invariant sets and that minimax procedures can be used to construct *nodal critical points* of the energy functional outside these invariant sets. As a byproduct we give more information of the dynamical nature of the gradient flow. This type of idea has been used successfully for elliptic problems on bounded domains (c.f. [1,4,9,13,14,18]). In general, the cone of positive (and negative) functions in the Sobolev space is invariant under the gradient flow. However, these cones have empty interior, and it is very difficult to build a deformation in relation to these cones and to construct critical points outside of these cones using a minimax method. In bounded domains, the dense subspace $C_0^1(\bar{\Omega})$ of $H_0^1(\Omega)$ has been used, since the cone of positive (and negative) functions in $C_0^1(\bar{\Omega})$ have nonempty interior. We remark also that an interesting approach without using the cones structure was used in [6] for bounded domain problems but may not be suitable for getting multiple nodal critical points for even functionals (see [5] for more references). For problems in $\mathbf{R}^N$, there is no known replacement for $C_0^1(\bar{\Omega})$ since $C_0^1(\mathbf{R}^N)$ has no interior points either. This has been the major obstacle for generalizing many results on nodal solutions from the bounded domain case to the entire space case. For a superlinear problem, this has been done recently [2]. However, it turns out that the existence of these invariant sets depend in a subtle way on the behavior of the functional near 0. In [2], under the condition that 0 is a strict local minimum critical point, it was shown that a neighborhood, in the Sobolev space norm, of the cone of positive (and negative) functions is an invariant set. When 0 is a saddle point (as in most of the cases of this paper) *no* neighborhood of the positive and negative cones can be invariant sets anymore. The main project of this paper is to develop new techniques to construct invariant sets for the case of 0 being a saddle point. Obviously, our construction can be used for more general type of problems.

The paper is organized as follows. Section 2 contains some preliminary technical results. Section 3 is devoted to the construction of a minimal positive solution and a maximal negative solution to (1.1). This is essential for the construction of invariant sets containing all positive and negative solutions to (1.1), so all the solutions obtained outside of these invariant sets by minimax procedures are nodal solutions. In Section 4, we construct new invariant sets which depend upon the local behavior of the energy functional I_λ near 0. Results in Sections 3 and 4 are also interesting for their own sake, and will be used in Section 5 for the proof of the existence results, Theorems 5.5 and 5.8, which together give Theorem 1.1.

2. Preliminaries

In the following, B_R will denote the ball in $\mathbf{R}^N$ centered at zero with radius R, $B_R^c := \mathbf{R}^N \backslash B_R$. The space $L^p(\mathbf{R}^N)$ will be denoted by L^p. We first state the following useful technical result:

Lemma 2.1 (*Van Heerden and Wang [25]*). *Assume* (V_1)–(V_2). *Then, for any* $\varepsilon > 0$ *there exist* $R > 0$ *and* $\Lambda > 0$ *such that*

$$\|u\|^2_{L^2(B_R^c)} \leqslant \varepsilon \|u\|^2_\lambda,$$

for all $u \in H_\lambda$ *and* $\lambda \geqslant \Lambda$.

For any given elements $\psi_1, \ldots, \psi_k$ of H_λ, we set

$$U_\lambda(\psi_1, \ldots, \psi_k) := \inf_{u \in H_\lambda} \{\|u\|^2_\lambda : \|u\|_{L^2} = 1, (u, \psi_i)_\lambda = 0 \text{ for } i = 1, \ldots, k\}.$$

For $k \in \mathbf{N}$, we define the spectral values of $-\Delta + V_\lambda$ by the kth Rayleigh quotient

$$\mu_k^\lambda = \sup_{\psi_1, \ldots, \psi_{k-1} \in H_\lambda} U_\lambda(\psi_1, \ldots, \psi_k).$$

For any $m \in \mathbf{N}$, [3, Corollary 2.2] asserts that for λ sufficiently large, the operator $-\Delta + V_\lambda$ has at least $d_m = \sum_{i=1}^m \dim(v_i)$ eigenvalues $\mu_1^\lambda, \ldots, \mu_{d_m}^\lambda$. The corresponding eigenfunctions are denoted by $e_1^\lambda, \ldots, e_{d_m}^\lambda$ with $\|e_k^\lambda\|_{L^2} = 1$. As a consequence of [24, Lemma 2.5], we conclude that:

Lemma 2.2. *Assume* (V_1)–(V_3). *Then* $\mu_k^\lambda \to \mu_k$ *as* $\lambda \to \infty$ *for all* $k \in \mathbf{N}$.

Proof. Fix any $k \in \mathbf{N}$. First we have $\mu_k^\lambda \leqslant \mu_k$ as a simple minimax description of the eigenvalues (see [20, Section XIII.1], for instance). According to [24, Lemma 2.5], the limit $\mu_k^* := \lim_{\lambda \to \infty} \mu_k^\lambda$ is an eigenvalue of (1.2). The proof of [24, Lemma 2.5] shows the weak limit $e_k := \lim_{\lambda \to \infty} e_k^\lambda$ in H_λ is an eigenfunction of (1.2) corresponding to μ_k^* and $\|e_k\|_{L^2} = 1$. Thus e_k^λ converges to e_k strongly in L^2 as $\lambda \to \infty$. For any $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $(e_i^\lambda, e_j^\lambda)_{L^2} = 0$ implies $(e_i, e_j)_{L^2(\Omega)} = 0$. As a consequence, $\mu_k^* \geqslant \mu_k$. This together with $\mu_k^\lambda \leqslant \mu_k$ gives the result. $\square$

Set

$$E_k := \text{span}\{e_1, \ldots, e_{d_k}\}, \quad E_0 := \{0\}$$

and

$$E_k(\lambda) := \text{span}\{e_1^\lambda, \ldots, e_{d_k}^\lambda\}, \quad E_0(\lambda) := \{0\}.$$

Let $E_k^\perp$ and $(E_k(\lambda))^\perp$ denote the orthogonal complement of E_k and $E_k(\lambda)$ in H_λ, respectively.

Recall that $(u_n) \in H_\lambda$ is a Palais–Smale ((PS) for short) sequence of I_λ if $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$. I_λ is said to satisfy the (PS)-condition if any such sequence contains a convergent subsequence. The following lemma is a more general version of a result in [24,25], and the proof here is different from, and simpler than that in [24,25].

Lemma 2.3. *Assume* (V_1)–(V_3), (f_0),(f_3) and (f_4). *Then there exists a* $\Lambda > 0$ *such that the functional* I_λ *satisfies the* (*PS*)*-condition for all* $\lambda \geqslant \Lambda$.

Proof. We only consider the case $n \geqslant 1$; the case $n = 0$ is similar and simpler. Take $v_n < b_1 \leqslant b_2 < v_{n+1}$ and $T > 0$ such that for $|u| \geqslant T$, $b_1 \leqslant f(x,u)/u \leqslant b_2$. By Lemma 2.2, we can take $\Lambda_1 > 0$ such that for $\lambda \geqslant \Lambda_1$

$$\min\left\{\frac{b_1}{\mu_{d_n}^\lambda} - 1,\ 1 - \frac{b_2}{\mu_{d_n+1}^\lambda}\right\} \geqslant a > 0$$

for some $a > 0$. By Lemma 2.1, there exist $\Lambda_2 > 0$ and $R_0 > 0$ such that for $\lambda \geqslant \Lambda_2$,

$$\|u\|^2_{L^2(B^c_{R_0})} \leqslant \varepsilon_0 \|u\|^2_\lambda, \quad u \in H_\lambda, \tag{2.1}$$

where

$$0 < \varepsilon_0 < \min\left\{\left(\frac{a}{2(\alpha + c_0)}\right)^2,\ 1\right\} \tag{2.2}$$

and $\alpha = b_1 b_2/(b_2 - b_1)$ and c_0 is from (f_4).

Now let (u_n) be a (PS) sequence for I_λ with $\lambda \geqslant \max\{\Lambda_1, \Lambda_2\}$. Let $\mathcal{Z}_n := \{x : |u_n(x)| \geqslant T\}$. Writing $u_n = v_n + w_n$ with $v_n \in E_n(\lambda)$ and $w_n \in (E_n(\lambda))^\perp$ and taking inner product of $I'_\lambda(u_n)$ and $v_n - w_n$, we see that

$$
\begin{aligned}
& o(1) \cdot \|u_n\|_\lambda \\
& \quad = \|v_n\|_\lambda^2 - \|w_n\|_\lambda^2 - \int_{\mathcal{Z}_n} \frac{f(x,\, u_n)}{u_n}(v_n^2 - w_n^2)\, dx - \int_{\mathcal{Z}_n^c} f(x,\, u_n)(v_n - w_n) \\
& \quad \leqslant \|v_n\|_\lambda^2 - \|w_n\|_\lambda^2 - b_1 \int_{\mathcal{Z}_n} v_n^2 + b_2 \int_{\mathcal{Z}_n} w_n^2 - \int_{\mathcal{Z}_n^c} f(x,\, u_n)(v_n - w_n) \\
& \quad = \|v_n\|_\lambda^2 - \|w_n\|_\lambda^2 - b_1 \int_{\mathbf{R}^N} v_n^2 + b_2 \int_{\mathbf{R}^N} w_n^2 + b_1 \int_{\mathcal{Z}_n^c} v_n^2 - b_2 \int_{\mathcal{Z}_n^c} w_n^2 \\
& \qquad - \int_{\mathcal{Z}_n^c} f(x,\, u_n)(v_n - w_n) \\
& \quad \leqslant -\left(\frac{b_1}{\mu_{d_n}^\lambda} - 1\right) \|v_n\|_\lambda^2 - \left(1 - \frac{b_2}{\mu_{d_n+1}^\lambda}\right) \|w_n\|_\lambda^2 + b_1 \int_{\mathcal{Z}_n^c} v_n^2 - b_2 \int_{\mathcal{Z}_n^c} w_n^2 \\
& \qquad - \int_{\mathcal{Z}_n^c} f(x,\, u_n)(v_n - w_n).
\end{aligned}
$$

Next, we claim

$$
b_1 \int_{\mathcal{Z}_n^c} v_n^2 - b_2 \int_{\mathcal{Z}_n^c} w_n^2 \leqslant \alpha \int_{\mathcal{Z}_n^c} u_n^2.
$$

To see this we choose $\beta = (\alpha - b_1)/\alpha = \alpha/(\alpha + b_2)$. Expanding $u_n^2 = v_n^2 + w_n^2 + 2v_n w_n$ and using Hölder's inequality, we get

$$
b_1 v_n^2 - b_2 w_n^2 - \alpha u_n^2 \leqslant (b_1 - \alpha + \alpha\beta) v_n^2 + (-\alpha + \alpha/\beta - b_2) w_n^2 = 0.
$$

On the other hand,

$$
\begin{aligned}
\left| \int_{\mathcal{Z}_n^c} f(x,\, u_n)(v_n - w_n) \right| & \leqslant \left(\int_{\mathcal{Z}_n^c} f(x, u_n)^2 \right)^{1/2} \left(\int_{\mathcal{Z}_n^c} (v_n - w_n)^2 \right)^{1/2} \\
& \leqslant c_0 \big(C(R_0) + \sqrt{\varepsilon_0} \|u_n\|_\lambda \big) \|u_n\|_\lambda.
\end{aligned}
$$

Combining these we get, with $C(R_0)$ a constant depending only on R_0,

$$
a\|u_n\|_\lambda^2 \leqslant o(1)\|u_n\|_\lambda + \alpha(C(R_0) + \varepsilon_0 \|u_n\|_\lambda^2) + c_0 C(R_0)\|u_n\|_\lambda + c_0 \sqrt{\varepsilon_0} \|u_n\|_\lambda^2,
$$

which gives the bound of the sequence. Assume that, without loss of any generality, $u_n \rightharpoonup u$ in H_λ and $u_n \to u$ in $L^2_{\rm loc}(\mathbf{R}^N)$ for some $u \in H_\lambda$, which is a solution to the problem

$$-\Delta u + V_\lambda(x)u = f(x, u) \quad \text{in } \mathbf{R}^N.$$

Taking inner product of $I'_\lambda(u_n)$ and $u_n - u$, noting that $I'_\lambda(u) = 0$, and using (f_4), we have for any $R \geqslant R_0$,

$$\begin{aligned}
\|u_n - u\|_\lambda^2 &= o(1) + \int_{\mathbf{R}^N} (f(x, u_n) - f(x, u))(u_n - u)\,dx \\
&\leqslant o(1) + c_0 \int_{B_R^c} (|u_n| + |u|)|u_n - u|\,dx \\
&\quad + \int_{B_R} |f(x, u_n) - f(x, u)||u_n - u|\,dx \\
&\leqslant o(1) + c_0 \int_{B_R^c} |u_n - u|^2\,dx + 2c_0 \int_{B_R^c} |u||u_n - u|\,dx \\
&\quad + \int_{B_R} |f(x, u_n) - f(x, u)||u_n - u|\,dx \\
&\leqslant o(1) + \frac{1}{2}\|u_n - u\|_\lambda^2 + 2c_0\|u_n - u\|_\lambda \left(\int_{B_R^c} |u|^2\,dx\right)^{1/2} \\
&\quad + \int_{B_R} |f(x, u_n) - f(x, u)||u_n - u|\,dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\|u_n - u\|_\lambda^2 \leqslant\ & o(1) + 4c_0\|u_n - u\|_\lambda \left(\int_{B_R^c} |u|^2\,dx\right)^{1/2} \\
& +2\int_{B_R} |f(x, u_n) - f(x, u)||u_n - u|\,dx.
\end{aligned}$$

Since

$$\lim_{R\to\infty} \left(\int_{B_R^c} |u|^2\,dx\right)^{1/2} = 0$$

and

$$\lim_{n\to\infty}\int_{B_R}|f(x,u_n)-f(x,u)||u_n-u|\,dx=0 \quad \text{for all } R>0,$$

we obtain $\lim_{n\to\infty}\|u_n-u\|_\lambda=0$, as required. $\square$

The following lemma is a variant of [18, Lemma 3.2]. It can be proved in a similar way as the proof of [18, Lemma 3.2] (cf. [18, Lemma 2.5]).

Lemma 2.4. *Let H be a Hilbert space, D_1 and D_2 be two closed convex subsets of H, and $I \in C^1(H,\mathbf{R})$. Suppose $I'(u)=u-A(u)$ and $A(D_i)\subset D_i$ for $i=1,\ 2$. Then there exists a pseudo gradient vector field V of I in the form $V(u)=u-B(u)$ with B satisfying $B(D_i)\subset D_i$ for $i=1,\ 2$. Moreover, $B(D_i)\subset \mathrm{int}(D_i)$ if $A(D_i)\subset \mathrm{int}(D_i)$ for $i=1,\ 2$, and V is odd if I is even and $D_1=-D_2$.*

Here recall that V is a pseudo gradient vector field of I if $V\in C(H,H)$, $V|_{H\setminus K}$ is locally Lipschitz continuous with $K:=\{u\in H : I'(u)=0\}$, and $(I'(u),V(u))\geqslant \frac{1}{2}||I'(u)||^2$ and $||V(u)||\leqslant 2||I'(u)||$ for all $u\in H$.

3. Minimal positive solution and maximal negative solution

In this section, we will construct a minimal positive solution and a maximal negative solution to problem (1.1). This is essential for the construction of invariant sets containing all the positive (and the negative) solutions in Section 4. In a bounded domain case, this can be done by an iterative procedure (see [17], for instance). Unfortunately, the argument used in bounded domain case does not work with our problem due to the lack of compactness of the operator. We will use variational arguments to construct these solutions.

With respect to the norm $\|\cdot\|_\lambda$, we have $I_\lambda'(u)=u-K_\lambda f(\cdot,u)$ where $K_\lambda : H_\lambda\to H_\lambda$ denotes the inverse operator of $-\Delta+V_\lambda$:

$$K_\lambda := \left(-\Delta+V_\lambda\right)^{-1}.$$

In view of (f_5), we may assume that $f(x,s)$ is increasing in s. Otherwise, we just replace the norm $\|\cdot\|_\lambda$ on H_λ with the equivalent one

$$\|u\|_{L,\lambda}^2 := \int_{\mathbf{R}^N}(|\nabla u|^2+(V_\lambda(x)+L)\,u^2)\,dx$$

and replace $f(x,u)$ with $f(x,u)+Lu$. What follows for the case $L=0$ work just as well for the case $L>0$. Throughout this section we will assume

($\tilde{f_2}$) $\liminf_{|s|\to 0} f(x,s)/s > \nu_1$ uniformly in x.

The main result of this section is as follows.

Theorem 3.1. *Assume* (V_1)–(V_3), (f_0), $(\tilde{f_2})$, (f_4), (f_5), *and assume that there exists a positive solution* w^+ *and a negative solution* w^- *to* (1.1). *Then there exists a minimal positive solution* u^+ *and a maximal negative solution* u^- *to* (1.1).

Proof. We only prove the existence of u^+. Define

$$\hat{f}(x,s)=\begin{cases} f(x,w^+(x)) & \text{if } s>w^+(x),\\ f(x,s) & \text{if } w^-(x)\leqslant s\leqslant w^+(x),\\ f(x,w^-(x)) & \text{if } s<w^-(x).\end{cases}$$

We consider the following cut-off problem:

$$\left.\begin{array}{l} -\Delta u+V_\lambda(x)u=\hat{f}(x,u) \quad \text{in } \mathbf{R}^N,\\ u\in H_\lambda.\end{array}\right\} \tag{3.1}$$

We claim that for all solutions u of (3.1), it holds $w^-(x)\leqslant u(x)\leqslant w^+(x)$ for all $x\in \mathbf{R}^N$. Seeking a contradiction, suppose $\mathcal{Z}:=\{x\in\mathbf{R}^N : u(x)>w^+(x)\}\neq\emptyset$ for some solution u of (3.1). Then $\mathcal{Z}$ is an open subset of $\mathbf{R}^N$ and on $\mathcal{Z}$:

$$-\Delta u+V_\lambda u=f(x,w^+)=-\Delta w^+ +V_\lambda w^+,$$

which implies that $-\Delta(u-w^+)+V_\lambda(u-w^+)=0$. Since $u(x),w^+(x)\to 0$ as $|x|\to\infty$, we conclude by the maximum principle that $u(x)=w^+(x)$ for all $x\in\mathcal{Z}$, a contradiction. In a similar fashion, we see that $w^-(x)\leqslant u(x)$ for all $x\in\mathbf{R}^N$.

Now consider the energy functional associated with (3.1):

$$\hat{I}_\lambda(u):=\frac{1}{2}\|u\|_\lambda^2-\int_{\mathbf{R}^N}\hat{F}(x,u)\,dx,$$

where $\hat{F}(x,u)=\int_0^u\hat{f}(x,s)\,ds$. The above discussion shows that any critical point of $\hat{I}_\lambda$ is a (weak) solution of the original problem (1.1). From the definition of $\hat{I}_\lambda$ and $\hat{f}$, it follows that

$$\hat{I}_\lambda(u)\geqslant\frac{1}{2}\|u\|_\lambda^2-C_1\int_{\mathbf{R}^N}\left(|w^+|+|w^-|\right)|u|\,dx\geqslant\frac{1}{2}\|u\|_\lambda^2-C_1\left(\|w^+\|_{L^2}+\|w^-\|_{L^2}\right)\|u\|_\lambda.$$

Here and in the sequel, C_i are constants. Thus $\hat{I}_\lambda$ is coercive. Next we verify the (PS)-condition for $\hat{I}_\lambda$. Suppose $(u_n) \in H_\lambda$ satisfy $\hat{I}_\lambda(u_n) \to c$ for some number c, and $\hat{I}'_\lambda(u_n) \to 0$. Then (u_n) is bounded in H_λ. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_λ and $u_n \to u$ in $L^2_{\text{loc}}(\mathbf{R}^N)$ with u a solution of (1.1). To obtain the strong convergence in H_λ, we observe that

$$\hat{I}'_\lambda(u_n)u_n = \left(\|u_n\|^2_\lambda - \|u\|^2_\lambda\right) - \int_{\mathbf{R}^N} \left(\hat{f}(x, u_n)u_n - \hat{f}(x, u)u\right) dx$$

and for any $R > 0$,

$$\begin{aligned} \left| \int_{B^c_R} \left(\hat{f}(x, u_n)u_n - \hat{f}(x, u)u\right) dx \right| &\leqslant C_2 \int_{B^c_R} \left(|w^+| + |w^-|\right)(|u_n| + |u|)\, dx \\ &\leqslant C_3 \left(\|w^+\|_{L^2(B^c_R)} + \|w^-\|_{L^2(B^c_R)}\right). \end{aligned}$$

Thus $\|u_n\|_\lambda \to \|u\|_\lambda$ which implies the strong convergence.

Recall that e_1 is the first eigenfunction of (1.2). We may assume $e_1 > 0$. By $(\tilde{f}_2)$, there exist $s^* > 0$ and $\delta > 0$ such that $f(x, s)/s > v_1 + \delta$ for $0 < |s| \leqslant s^*$. Choose $R_1 > 0$ and $s_0 > 0$ such that $w^+(x) < s^*$ for $|x| \geqslant R_1$ and $s_0 e_1(x) < w^+(x)$ for $|x| \leqslant R_1$. We claim that $s_0 e_1(x) < w^+(x)$ for all $x \in \mathbf{R}^N$. Indeed, if $\mathcal{X} := \left\{x \in \mathbf{R}^N : s_0 e_1(x) > w^+(x)\right\} \neq \emptyset$, then $\mathcal{X} \subset B^c_{R_1} \cap \Omega$ and we have

$$-\Delta(s_0 e_1) + V_\lambda s_0 e_1 = v_1 s_0 e_1 \quad \text{in } \mathcal{X}, \tag{3.2}$$

$$-\Delta w^+ + V_\lambda w^+ = f(x, w^+) \quad \text{in } \mathcal{X}, \tag{3.3}$$

$$s_0 e_1 = w^+ \quad \text{on } \partial\mathcal{X}, \tag{3.4}$$

$$s_0 \frac{\partial e_1}{\partial v} \leqslant \frac{\partial w^+}{\partial v} \text{ on } \partial\mathcal{X}, \tag{3.5}$$

where v denotes the outer unit normal to $\partial\mathcal{X}$. Multiplying (3.3) with $(s_0 e_1)$ and subtracting it from (3.2) multiplied with w^+ yields

$$\int_{\mathcal{X}} \left((s_0 e_1)\Delta w^+ - w^+ \Delta(s_0 e_1)\right) dx = \int_{\mathcal{X}} s_0 e_1 \left(v_1 w^+ - f(x, w^+)\right) dx.$$

Since for $x \in \mathcal{X}$, $f(x, w^+) > (v_1 + \delta)w^+$, we see that

$$\int_{\mathcal{X}} s_0 e_1 \left(v_1 w^+ - f(x, w^+)\right) dx < \int_{\mathcal{X}} (s_0 e_1)(-\delta w^+)\, dx < 0.$$

However, (3.4), (3.5) and the divergence theorem imply

$$\int_{\mathcal{X}} \left((s_0e_1)\Delta w^+ - w^+\Delta(s_0e_1)\right)\,dx = \int_{\partial\mathcal{X}} s_0e_1\left(\frac{\partial w^+}{\partial\nu} - s_0\frac{\partial e_1}{\partial\nu}\right)d\sigma \geqslant 0,$$

a contradiction. Thus, for $0 < s \leqslant s_0$,

$$\hat{I}_\lambda(se_1) = I_\lambda(se_1) = \frac{\nu_1 s^2}{2}\int_\Omega e_1^2\,dx - \int_\Omega F(x, se_1)\,dx.$$

Decreasing s_0 if necessary, we may assume $F(x, se_1(x)) > (\nu_1+\delta)s^2e_1^2(x)/2$ for all $0 < s \leqslant s_0$ and $x \in \mathbf{R}^N$. Then,

$$\hat{I}_\lambda(se_1) < \frac{\nu_1 s^2}{2}\int_\Omega e_1^2\,dx - \frac{(\nu_1+\delta)s^2}{2}\int_\Omega e_1^2\,dx < 0,$$

for all $0 < s \leqslant s_0$.

The fact that 0 and w^+ are solutions of (1.1) and f is increasing in s imply that $0 \leqslant K_\lambda f(x,u) \leqslant w^+$ if $0 \leqslant u \leqslant w^+$. For $0 < s \leqslant s_0$ and $t \geqslant 0$, consider the following initial value problem:

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\eta_s^t &= -V(\eta_s^t), \\ \eta_s^0 &= se_1, \end{aligned}\right\} \tag{3.6}$$

where V is a pseudo gradient vector field of $\hat{I}_\lambda$ in the form $V = I - B$ with B satisfying $B([0,\ w^+]) \subset [0,\ w^+]$ and $[0,\ w^+] = \{u \in H_\lambda:\ 0 \leqslant u \leqslant w^+\}$; the existence of V is guaranteed by Lemma 2.4 in which $D_1 = [0,\ w^+]$ and $D_2 = \emptyset$. Let $[0,\ T_s)$ denote the existence interval of (3.6). Since $\hat{I}_\lambda$ is coercive and satisfy the (PS)-condition, a standard argument shows that there exists a sequence $t_n^s \to T_s$ such that $\eta_s^{t_n^s} \to u^s$ for some $u^s \in H_\lambda$ satisfying $I_\lambda(u^s) < 0$ and $I'_\lambda(u^s) = 0$. Since $B([0,\ w^+]) \subset [0,\ w^+]$ and $0 \leqslant se_1 \leqslant w^+$, a theorem of Brezis–Martin [7, Chapter 1, Theorem 6.3] implies that $0 \leqslant \eta_s^t(x) \leqslant w^+(x)$ for all $x \in \mathbf{R}^N$, $0 \leqslant t < T_s$ and $0 < s \leqslant s_0$. Thus $0 < u^{s_0}(x) \leqslant w^+(x)$ for all $x \in \mathbf{R}^N$. Choose $s_1 \in (0, s_0/2)$ such that $0 \leqslant s_1e_1(x) \leqslant u^{s_0}(x)$ for all $x \in \mathbf{R}^N$. Then the above discussion implies $0 \leqslant \eta_{s_1}^t(x) \leqslant u^{s_0}(x)$ for all $t \in [0,\ T_{s_1})$ and there exists a positive solution u^{s_1} of (1.1) in the ω-limit set of $\eta_{s_1}^t$ satisfying $0 < u^{s_1}(x) \leqslant u^{s_0}(x)$ for all $x \in \mathbf{R}^N$. Here a different pseudo gradient vector field $V = I - B$ may be used in order that $B([0,\ u^{s_0}]) \subset [0,\ u^{s_0}]$. Repeating the above process infinitely many times, we obtain a decreasing sequence of positive numbers (s_n) and positive solutions (u^{s_n}) of (1.1) such that $s_n \to 0$, $\hat{I}_\lambda(u^{s_n}) < 0$,

$$0 < \ldots \leqslant u^{s_n}(x) \leqslant u^{s_{n-1}} \leqslant \cdots \leqslant u^{s_0}(x) \leqslant w^+(x) \quad \text{for all } x \in \mathbf{R}^N$$

and u^{s_n} lies in the ω-limit set of $\eta^t_{s_n}$. Obviously, (u^{s_n}) is a (PS)-sequence. The (PS)-condition, together with the monotonicity of (u^{s_n}) imply that $u^{s_n} \to u^+$ in H_λ and $u^{s_n}(x) \to u^+(x)$ for all $x \in \mathbf{R}^N$ and for some nonnegative solution u^+ of Eq. (1.1).

Now we prove that u^+ is a positive solution of (1.1). Arguing indirectly, we assume $u^+ = 0$. Then

$$u^{s_n} \to 0 \quad \text{in } H_\lambda. \tag{3.7}$$

Choose a natural number k such that $0 < N - 2 - 3k < 4$. Set $\gamma_i = 1 + i/k$ for $i = 0, 1, \dots, k$. For any $R > 0$, from the elliptic L^p_{loc} estimates [10, Theorem 9.11], we have

$$\begin{aligned}
&\|u^{s_n}\|_{W^{2,2N/(N-2)}(B_{\gamma_{k-1}R})} \\
&\quad \leqslant C_4 \left(\|f(x, u^{s_n})\|_{L^{2N/(N-2)}(B_{\gamma_k R})} + \|u^{s_n}\|_{L^{2N/(N-2)}(B_{\gamma_k R})} \right) \\
&\quad \leqslant C_5 \|u^{s_n}\|_{L^{2N/(N-2)}(B_{\gamma_k R})} \leqslant C_6 \|u^{s_n}\|_\lambda,
\end{aligned}$$

which implies

$$\|u^{s_n}\|_{L^{2N/(N-5)}(B_{\gamma_{k-1}R})} \leqslant C_7 \|u^{s_n}\|_\lambda,$$

where C_7 depends only on N, R and f, but is independent of n. Repeating the above k times, we obtain,

$$\|u^{s_n}\|_{W^{2,2N/(N-2-3k)}(B_R)} \leqslant C_8 \|u^{s_n}\|_\lambda.$$

Since $2N/(N-2-3k) > N/2$, the embedding $W^{2,2N/(N-2-3k)}(B_R) \hookrightarrow C(B_R)$ is continuous. Thus

$$\|u^{s_n}\|_{L^\infty(B_R)} \leqslant C_9 \|u^{s_n}\|_\lambda. \tag{3.8}$$

Since u^{s_0} solves (1.1), for the number s^* given above there exists a $R_2 > 0$ such that $u^{s_0} < s^*$ for all $|x| \geqslant R_2$. From (3.7) and (3.8) there exists an $N^* > 0$ such that $u^{s_n}(x) < s^*$ for all $|x| \leqslant R_2$ and $n \geqslant N^*$. So,

$$u^{s_n}(x) < s^* \quad \text{for all } x \in \mathbf{R}^N \text{ and } n \geqslant N^*.$$

Let $e_1^\lambda > 0$ denote the first eigenfunction of the operator $-\Delta + V_\lambda$, i.e.

$$-\Delta e_1^\lambda + V_\lambda e_1^\lambda = \mu_1^\lambda e_1^\lambda \quad \text{on } \mathbf{R}^N.$$

Recall that $\mu_1^\lambda \leqslant \nu_1$ for all $\lambda \geqslant 0$. Finally, multiplying

$$-\Delta u^{s_n} + V_\lambda u^{s_n} = f(x, u^{s_n})$$

with e_1^λ and integrating, we obtain (for $n \geqslant N^*$),

$$\mu_1^\lambda \int_{\mathbf{R}^N} u^{s_n} e_1^\lambda \, dx = \int_{\mathbf{R}^N} f(x, u^{s_n}) e_1^\lambda \, dx \geqslant (\mu_1^\lambda + \delta) \int_{\mathbf{R}^N} u^{s_n} e_1^\lambda \, dx,$$

a contradiction.

To complete the proof, it remains to show that u^+ constructed above is indeed a minimal positive solution. Since $u^+(x) > 0$ on $\mathbf{R}^N$, from the above discussion we can choose an $N_1 > 0$ such that $s_n e_1(x) \leqslant u^+(x)$ for all $n \geqslant N_1$ and $x \in \mathbf{R}^N$. Thus $\eta_{s_n}^t(x) \leqslant u^+(x)$ for all $n \geqslant N_1$, $t \geqslant 0$ and $x \in \mathbf{R}^N$, which implies

$$u^{s_n}(x) \leqslant u^+(x) \quad \text{for all } n \geqslant N_1 \text{ and } x \in \mathbf{R}^N.$$

Since $(u^{s_n}(x))$ is decreasing in n and $u^{s_n}(x) \to u^+(x)$ for all $x \in \mathbf{R}^N$,

$$u^{s_n}(x) = u^+(x) \quad \text{for all } n \geqslant N_1 \text{ and } x \in \mathbf{R}^N.$$

Now, let u_1 be another positive solution of (1.1). Fixing an $n_1 \geqslant N_1$ such that $s_{n_1} e_1(x) \leqslant u_1(x)$, we have $\eta_{s_{n_1}}^t(x) \leqslant u_1(x)$ for all $t \geqslant 0$ and therefore

$$u^+(x) = u^{s_{n_1}}(x) \leqslant u_1(x) \quad \text{for all } x \in \mathbf{R}^N.$$

This shows that u^+ is the minimal positive solution of problem (1.1). $\square$

Remark 3.2. We note that the above proof is independent of λ. Thus the conclusion of Theorem 3.1 holds true for a large class of potentials. For instance, we can obtain the following more general result: Assume $V \in C(\mathbf{R}^N, \mathbf{R})$, V is bounded below, and f satisfies (f_0), (f_4), and (f_5). Assume in addition that uniformly in x

$$\liminf_{s \to 0} \frac{f(x,\ s)}{s} > \mu_1,$$

where

$$\mu_1 := \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\int_{\mathbf{R}^N} (|\nabla u|^2 + V u^2)\, dx}{\int_{\mathbf{R}^N} u^2 \, dx}.$$

Then the problem

$$-\Delta u + V(x)u = f(x,u), \quad u \in H_0^1(\mathbf{R}^N)$$

has a minimal positive (maximal negative) solution provided that it has a positive (negative) solution.

We close this section with a sufficient condition for the existence of a positive (and negative) solution to (1.1). From Theorem 3.1 this would imply the existence of a minimal positive (maximal negative) solution.

Theorem 3.3. *Assume* (V_1)–(V_3), (f_0), $(\tilde{f}_2)$, (f_4) *and uniformly in* $x \in \mathbf{R}^N$

$$\limsup_{|s|\to\infty} \frac{f(x,s)}{s} < v_1.$$

Then, for λ *sufficiently large, problem* (1.1) *has a positive and a negative solution.*

Proof. We only prove the existence of a positive solution. We first show that, for λ sufficiently large, I_λ is coercive. By assumption, there exist a $\delta > 0$ and $C_1 > 0$ such that

$$F(x,s) \leqslant \frac{1}{2}(v_1 - \delta)s^2 + C_1 \quad \text{for all } s \in \mathbf{R}.$$

For any $R > 0$,

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{B_R} F(x,u)\,dx - \int_{B_R^c} F(x,u)\,dx \\
&\geqslant \frac{1}{2}\|u\|_\lambda^2 - \frac{v_1 - \delta}{2}\int_{B_R} u^2\,dx - C_1|B_R| - C_2\int_{B_R^c} u^2\,dx \\
&\geqslant \frac{1}{2}\|u\|_\lambda^2 - \frac{v_1 - \delta}{2\mu_1^\lambda}\|u\|_\lambda^2 - C_1|B_R| - C_2\int_{B_R^c} u^2\,dx \\
&\geqslant \frac{\delta}{4v_1}\|u\|_\lambda^2 - C_1|B_R| - C_2\int_{B_R^c} u^2\,dx,
\end{aligned}$$

for λ sufficiently large, say $\lambda \geqslant \Lambda_0$ (Here we used Lemma 2.2). From Lemma 2.1, there exist $R_1 > 0$ and $\Lambda_1 \geqslant \Lambda_0$ such that

$$\int_{B_{R_1}^c} u^2\,dx \leqslant \frac{\delta}{8C_2 v_1}\|u\|_\lambda^2,$$

for all $\lambda \geqslant \Lambda_1$. It follows that

$$I_\lambda(u) \geqslant \frac{\delta}{8\nu_1}\|u\|_\lambda^2 - C_1\left|B_{R_1}\right|,$$

for all $\lambda \geqslant \Lambda_1$, i.e. I_λ is coercive.

Fix an s such that $I_\lambda(se_1) < 0$, the existence of such an s was showed in the proof of Theorem 3.1. Consider the initial value problem

$$\left.\begin{aligned} \tfrac{d}{dt}\eta^t &= -V(\eta^t), \\ \eta^0 &= se_1. \end{aligned}\right\}$$

Here V is a pseudo gradient vector field of I_λ in the form $V = I - B$ with B satisfying $B(P_\lambda^+) \subset P_\lambda^+$, where $P_\lambda^+ := \{u \in H_\lambda :\ u \geqslant 0\}$ is the positive cone in H_λ. The existence of such a V is guaranteed by (f_4) and by Lemma 2.4 (by (f_4), we may assume that $f(x,s) \geqslant 0$ for all $s \geqslant 0$, otherwise we need just to replace V_λ and $f(x,s)$ with $V_\lambda + c_0$ and $f(x,s) + c_0 s$, respectively). Since I_λ is coercive, a standard argument shows that there exists a positive sequence $t_n \to \infty$ such that

$$I_\lambda(\eta^{t_n}) \ \leqslant\ I_\lambda(se_1) < 0 \quad \text{for all } n, \tag{3.9}$$

$$I'_\lambda(\eta^{t_n}) \ \to\ 0. \tag{3.10}$$

According to Lemma 2.3, passing to a subsequence if necessary we may assume $\eta^{t_n} \to u$ in H_λ, where u is a solution of (1.1). Since $B(P_\lambda^+) \subset P_\lambda^+$ and $se_1 \in P_\lambda^+$, $\eta^{t_n} \in P_\lambda^+$. Therefore $u \in P_\lambda^+$. The fact that $I_\lambda(u) \leqslant I_\lambda(\eta^{t_n}) \leqslant I_\lambda(se_1) < 0$ implies $u \neq 0$. Then by the maximum principle, $u(x) > 0$, for all $x \in \mathbf{R}^N$.

Remark 3.4. Under stronger assumptions, the existence of a positive and a negative solution was obtained in [24].

4. Invariant sets of the gradient flow

In order to construct nodal solutions we need to isolate the signed solutions (positive and negative solutions) into certain invariant sets. This section is devoted to this purpose. Here we discover that it is the behavior of I_λ near the trivial critical point 0 that plays an important role for the structure of these invariant sets. This will be done by distinguishing the two opposite cases: $\limsup_{|s|\to 0} f(x,s)/s < \nu_1$ and $\liminf_{|s|\to 0} f(x,s)/s > \nu_1$. Define $A(u) := K_\lambda f(\cdot,u) = (-\Delta + V_\lambda)^{-1} f(\cdot,u)$, $u \in H_\lambda$ and

$$\left.\begin{aligned} \tfrac{d}{dt}\eta^t(u) &= -\eta^t(u) + B(\eta^t(u)) \\ \eta^0(u) &= u, \end{aligned}\right\}$$

where B is related to A via Lemma 2.4 in which D_1 and D_2 will be constructed in Theorem 4.1 and Theorem 4.2. This section is concerned with the construction of these sets which are invariant under the flow $\eta^t(u)$ such that all positive (and negative) solutions are contained in these invariant sets. Recall that a subset $W \subset H_\lambda$ is an invariant set with respect to η if, for any $u \in W$, $\eta^t(u) \in W$ for all $t > 0$.

Denote

$$P_\lambda^\pm = \{u \in H_\lambda : \pm u \geqslant 0\}.$$

For any $M \subset H_\lambda$ and $\varepsilon > 0$, M_ε denotes the closed ε-neighborhood of M, i.e.

$$M_\varepsilon := \{u \in H_\lambda : \mathrm{dist}_\lambda(u, M) \leqslant \varepsilon\}.$$

The following result shows that a neighborhood of $P_\lambda^\pm$ is an invariant set if 0 is a local minimum critical point of the functional. This result was proved in [2] for a superlinear problem, but the proof covers our case as well. We quote the proof from [2] for the readers convenience. We also note that this result is essentially λ-independent as long as $f_s'(x, 0) < \mu_1^\lambda$.

Theorem 4.1 (*Bartsch et al. [2]*). *Assume* (V_1)–(V_3), (f_0), (f_4), *and* (f_5). *If* $\limsup_{|s|\to 0} f(x,s)/s < \nu_1$ *then there exist* $\varepsilon_0 > 0$ *and* $\Lambda > 0$ *such that*

$$A((P_\lambda^\pm)_\varepsilon) \subset \mathrm{int}((P_\lambda^\pm)_\varepsilon) \quad for\ all\ 0 < \varepsilon \leqslant \varepsilon_0,\ \ \lambda \geqslant \Lambda,$$

and

$$\eta^t((P_\lambda^\pm)_\varepsilon) \subset \mathrm{int}((P_\lambda^\pm)_\varepsilon) \quad for\ all\ t > 0,\ 0 < \varepsilon \leqslant \varepsilon_0,\ and\ \lambda \geqslant \Lambda.$$

Proof. For $u \in H_\lambda$, we denote $v = Au$ and $u^+ = \max\{0, u\}$, $u^- = \min\{0, u\}$. Note that for any $u \in H_\lambda$ and $2 \leqslant p \leqslant 2^* := 2N/(N-2)$,

$$\|u^-\|_{L^p} = \inf_{w \in P_\lambda^+} \|u - w\|_{L^p}. \tag{4.1}$$

Since,

$$\|v^-\|_\lambda^2 = (v, v^-)_\lambda = \int_{\mathbf{R}^N} (\nabla v \cdot \nabla v^- + V_\lambda v v^-)\, dx = \int_{\mathbf{R}^N} f(x, u) v^-\, dx,$$

the fact that $v^+ \in P_\lambda^+$ and $v - v^+ = v^-$, implies

$$\operatorname{dist}_\lambda(v, P_\lambda^+) \cdot \|v^-\|_\lambda \leqslant \|v^-\|_\lambda^2 \leqslant \int_{\mathbf{R}^N} f(x, u^-)v^- \, dx. \tag{4.2}$$

Under the assumption $\limsup_{|s|\to 0} f(x,s)/s < v_1$ there exist a $\delta > 0$ and $C_1 > 0$ such that

$$f(x,s) \geqslant (v_1 - \delta)s + C_1|s|^{2^*-2}s \quad \text{for } s \leqslant 0.$$

Thus

$$\begin{aligned}\int_{\mathbf{R}^N} f(x,u^-)v^- \, dx &\leqslant \int_{\mathbf{R}^N} \left[(v_1-\delta)u^- + C_1|u^-|^{2^*-2}u^-\right] v^- \, dx \\ &\leqslant (v_1-\delta)\|u^-\|_{L^2}\|v^-\|_{L^2} + C_1\|u^-\|_{L^{2^*-1}}^{2^*-1}\|v^-\|_{L^{2^*-1}}. \end{aligned} \tag{4.3}$$

From the Sobolev imbedding and (4.1)–(4.3),

$$\operatorname{dist}_\lambda(v, P_\lambda^+) \cdot \|v^-\|_\lambda \leqslant \frac{v_1-\delta}{\sqrt{\mu_1^\lambda}} \inf_{w \in P_\lambda^+} \|u-w\|_{L^2}\|v^-\|_\lambda + C_2 \inf_{w\in P_\lambda^+} \|u-w\|_{L^{2^*-1}}^{2^*-1}\|v^-\|_\lambda,$$

which implies (if $\|v^-\|_\lambda \neq 0$),

$$\begin{aligned}\operatorname{dist}_\lambda(v, P_\lambda^+) &\leqslant \frac{v_1-\delta}{\sqrt{\mu_1^\lambda}} \inf_{w\in P_\lambda^+} \|u-w\|_{L^2} + C_2 \inf_{w\in P_\lambda^+}\|u-w\|_{L^{2^*-1}}^{2^*-1} \\ &\leqslant \frac{v_1-\delta}{\mu_1^\lambda} \inf_{w\in P_\lambda^+}\|u-w\|_\lambda + C_3 \inf_{w\in P_\lambda^+}\|u-w\|_\lambda^{2^*-1} \\ &= \frac{v_1-\delta}{\mu_1^\lambda}\operatorname{dist}_\lambda(u, P_\lambda^+) + C_3\left(\operatorname{dist}_\lambda(u, P_\lambda^+)\right)^{2^*-1}.\end{aligned}$$

Therefore, there exist $\varepsilon_0 > 0$ and $\Lambda > 0$ such that if $\operatorname{dist}_\lambda(u, P_\lambda^+) \leqslant \varepsilon_0$ and $\lambda \geqslant \Lambda$ then

$$\operatorname{dist}_\lambda(v, P_\lambda^+) < \operatorname{dist}_\lambda(u, P_\lambda^+).$$

The first conclusion in Theorem 4.1 is proved. The second conclusion is a consequence of the first as shown in [18] via Lemma 2.4. □

Next, we consider the case $\liminf_{|s|\to 0} f(x,s)/s > v_1$. We first note that any neighborhoods of the positive (and negative) cones are no longer invariant sets of the gradient flow. We give a new construction here. Choose $\delta > 0$ and $s_0 > 0$ such that

$$\frac{f(x,s)}{s} > \nu_1 + \delta \geqslant \mu_1^\lambda + \delta, \tag{4.4}$$

for $0 < s \leqslant s_0$ and $x \in \mathbf{R}^N$. Let $e_1^\lambda \in H_\lambda$ be the eigenfunction associated with the eigenvalue μ_1^λ (we assume λ to be sufficiently large) such that $e_1^\lambda > 0$ and $\max_{x \in \mathbf{R}^N} e_1^\lambda \leqslant s_0$. According to Theorem 3.1 and its proof, we may also assume that

$$w^-(x) \leqslant -e_1^\lambda(x) \quad \text{and } e_1^\lambda(x) \leqslant w^+(x),$$

for all $x \in \mathbf{R}^N$, all negative solutions w^- and positive solutions w^+ of (1.1). Define

$$D_\lambda^\pm := \{u \in H_\lambda : \pm u \geqslant e_1^\lambda\}.$$

From the above discussion, all positive solutions and negative solutions to (1.1) are contained in D_λ^+ and D_λ^-, respectively. We show in the following that under the condition $\liminf_{|s|\to 0} f(x,s)/s > \nu_1$, suitable neighborhoods of these sets are invariant sets. This result is λ-dependent and holds only for λ large.

Theorem 4.2. *Assume* $(V_1)-(V_3)$, (f_0), (f_4), *and* (f_5). *If* $\liminf_{|s|\to 0} f(x,s)/s > \nu_1$, *then there exist* $\varepsilon_0 > 0$ *and* $\Lambda > 0$ *such that*

$$A((D_\lambda^\pm)_\varepsilon) \subset \operatorname{int}((D_\lambda^\pm)_\varepsilon) \quad \textit{for all } 0 < \varepsilon \leqslant \varepsilon_0, \ \lambda \geqslant \Lambda,$$

and

$$\eta^t((D_\lambda^\pm)_\varepsilon) \subset \operatorname{int}((D_\lambda^\pm)_\varepsilon) \quad \textit{for all } t > 0, \ 0 < \varepsilon \leqslant \varepsilon_0, \ \textit{and } \lambda \geqslant \Lambda.$$

Proof. We only prove the result for the positive sign, the other case follows analogously. For $u \in H_\lambda$ we denote

$$v = Au \quad \text{and} \quad v_1 = \max\{e_1^\lambda, v\}.$$

Then $\operatorname{dist}_\lambda(v, D_\lambda^+) \leqslant \|v - v_1\|_\lambda$ which implies

$$\operatorname{dist}_\lambda(v, D_\lambda^+) \cdot \|v - v_1\|_\lambda \leqslant \|v - v_1\|_\lambda^2.$$

Note that $v_1 = e_1^\lambda$ if $v_1 \neq v$. Since

$$\begin{aligned}\|v - v_1\|_\lambda^2 &= (v - e_1^\lambda, v - v_1)_\lambda \\ &= \int_{\mathbf{R}^N} \nabla(v - e_1^\lambda)\nabla(v - v_1) + V_\lambda(v - e_1^\lambda)(v - v_1)\,dx \\ &= \int_{\mathbf{R}^N} \left(-\Delta(v - e_1^\lambda) + V_\lambda(v - e_1^\lambda)\right)(v - v_1)\,dx \\ &= \int_{\mathbf{R}^N} \left(f(x,u) - \mu_1^\lambda e_1^\lambda\right)(v - v_1)\,dx,\end{aligned}$$

we have

$$\mathrm{dist}_\lambda(v, D_\lambda^+)\cdot\|v - v_1\|_\lambda \leqslant \int_{\mathbf{R}^N}\left(\mu_1^\lambda e_1^\lambda - f(x,u)\right)(v_1 - v)\,dx.$$

For the number s_0 given in (4.4), if $u > s_0$, then

$$f(x,u) \geqslant f(x,s_0) > (\mu_1^\lambda + \delta)s_0 \geqslant (\mu_1^\lambda + \delta)e_1^\lambda.$$

Therefore, since $v \leqslant v_1$, we conclude that

$$\mathrm{dist}_\lambda(v, D_\lambda^+)\cdot\|v - v_1\|_\lambda \leqslant \int_{u(x)\leqslant s_0}\left(\mu_1^\lambda e_1^\lambda - f(x,u)\right)(v_1 - v)\,dx.$$

Since $f(x,u) \geqslant (\mu_1^\lambda + \delta)u$ for $0 \leqslant u \leqslant s_0$ and $f(x,u) \geqslant c_0 u$ for $u \leqslant 0$,

$$\begin{aligned}&\mathrm{dist}_\lambda(v, D_\lambda^+)\cdot\|v - v_1\|_\lambda \\ &\quad\leqslant \int_{0\leqslant u(x)\leqslant s_0}\left(\mu_1^\lambda e_1^\lambda - (\mu_1^\lambda + \delta)u\right)(v_1 - v)\,dx + \int_{u(x)<0}\left(\mu_1^\lambda e_1^\lambda - c_0 u\right)(v_1 - v)\,dx \\ &\quad\triangleq I_1 + I_2.\end{aligned}$$

Note that

$$\begin{aligned}I_1 &\leqslant \int_{0\leqslant u(x)\leqslant \mu_1^\lambda e_1^\lambda(x)/(\mu_1^\lambda+\delta)}\left(\mu_1^\lambda e_1^\lambda - (\mu_1^\lambda + \delta)u\right)(v_1 - v)\,dx \\ &\leqslant (\mu_1^\lambda + \delta)\int_{0\leqslant u(x)\leqslant \mu_1^\lambda e_1^\lambda(x)/(\mu_1^\lambda+\delta)}(e_1^\lambda - u)(v_1 - v)\,dx\end{aligned}$$

and

$$I_2 \leqslant (\mu_1^\lambda + c_0) \int_{u(x)<0} (e_1^\lambda - u)(v_1 - v)\, dx.$$

Denoting $C = \max\{\mu_1^\lambda + \delta,\ \mu_1^\lambda + c_0\}$, we have

$$\operatorname{dist}_\lambda(v, D_\lambda^+) \cdot \|v - v_1\|_\lambda \leqslant C \int_{\Theta^\lambda} (e_1^\lambda - u)(v_1 - v)\, dx := C E^\lambda, \tag{4.5}$$

where

$$\Theta^\lambda := \left\{ x \in \mathbf{R}^N :\ u(x) \leqslant \frac{\mu_1^\lambda}{\mu_1^\lambda + \delta} e_1^\lambda(x) \right\}.$$

For any $R > 0$, we write

$$E^\lambda = \int_{\Theta^\lambda \cap B_R} (e_1^\lambda - u) \cdot (v_1 - v)\, dx + \int_{\Theta^\lambda \cap B_R^c} (e_1^\lambda - u) \cdot (v_1 - v)\, dx := E_1^\lambda + E_2^\lambda.$$

Since, on $\Theta^\lambda \cap B_R$,

$$e_1^\lambda - u \geqslant \frac{\delta}{\mu_1^\lambda + \delta} e_1^\lambda \geqslant \delta_R$$

for some $\delta_R > 0$, there exists a $C_1 = C_1(R) > 0$ such that

$$\begin{aligned} E_1^\lambda &\leqslant C_1(R) \int_{\Theta^\lambda \cap B_R} |e_1^\lambda - u|^{2^*-1} |v_1 - v|\, dx \\ &\leqslant C_1(R) \|e_1^\lambda - u\|_{L^{2^*}(\Theta^\lambda \cap B_R)}^{2^*-1} \|v_1 - v\|_{L^{2^*}} \\ &\leqslant C_1(R) \|e_1^\lambda - u\|_{L^{2^*}(\Theta^\lambda \cap B_R)}^{2^*-1} \|v - v_1\|_\lambda. \end{aligned} \tag{4.6}$$

The fact $u(x) \leqslant e_1^\lambda(x)$ on Θ^λ implies

$$\|e_1^\lambda - u\|_{L^p(\Theta^\lambda)} = \inf_{w \in D_\lambda^+} \|w - u\|_{L^p(\Theta^\lambda)},$$

for all $u \in H_\lambda$ and $2 \leqslant p \leqslant 2^*$. Then, by the Sobolev imbedding and (4.6),

$$E_1^\lambda \leqslant C_2(R) \left(\operatorname{dist}_\lambda(u, D_\lambda^+) \right)^{2^*-1} \cdot \|v_1 - v\|_\lambda. \tag{4.7}$$

From Lemma 2.1, there exist $R_0 > 0$ and $\Lambda_0 > 0$ such that

$$\begin{aligned} E_2^\lambda &\leqslant \|e_1^\lambda - u\|_{L^2(\Theta^\lambda \cap B_{R_0}^c)} \|v_1 - v\|_{L^2} \\ &\leqslant \inf_{w \in D_\lambda^+} \|u - w\|_{L^2(\Theta^\lambda \cap B_{R_0}^c)} \|v_1 - v\|_\lambda \\ &\leqslant \frac{1}{2C} \operatorname{dist}_\lambda(u, D_\lambda^+) \cdot \|v_1 - v\|_\lambda, \end{aligned} \tag{4.8}$$

for all $\lambda \geqslant \Lambda_0$, where C is the number from (4.5). Combining (4.5)–(4.8) we conclude that

$$\operatorname{dist}_\lambda(v, D_\lambda^+) \leqslant C \left(\frac{1}{2C} \operatorname{dist}_\lambda(u, D_\lambda^+) + C_2(R_0) \left(\operatorname{dist}_\lambda(u, D_\lambda^+) \right)^{2^*-1} \right),$$

for all $v \notin D_\lambda^+$ and $\lambda \geqslant \Lambda_0$. Therefore there exists an $\varepsilon_0 > 0$ such that if $\operatorname{dist}_\lambda(u, D_\lambda^+) \leqslant \varepsilon_0$, it holds

$$\operatorname{dist}_\lambda(Au, D_\lambda^+) = \operatorname{dist}_\lambda(v, D_\lambda^+) < \operatorname{dist}_\lambda(u, D_\lambda^+).$$

Once again, the second conclusion follows directly from [18], via Lemma 2.4. $\square$

5. Proof of Theorem 1.1

In order to prove our main result Theorem 1.1, we need to use Ljusternik–Schnirelman type minimax results. In general, results of this type do not give the nodal information of the solutions. We need to have the order structure and the invariant sets of the gradient flow from the previous section built into the minimax arguments. We need to distinguish two cases: $d_m > d_n$ and $d_n > d_m$. Though the results are similar for the two cases, the minimax arguments used are quite different and the structure of the invariant sets are also quite different. These two cases will be dealt with in Sections 5.2 and 5.3, with some technical preparations in Section 5.1.

5.1. A deformation lemma in the presence of invariant sets

Let us start with a more abstract setting. Consider $I \in C^1(X, \mathbf{R})$ where X is a Banach space. Let V be a pseudo gradient vector field of I such that V is odd if I is even, and consider

$$\left.\begin{aligned} \tfrac{d}{dt}\sigma(t,u) &= -V(\sigma), \\ \sigma(0,u) &= u \in X. \end{aligned}\right\} \tag{5.1}$$

A subset $W \subset X$ is an invariant set with respect to σ if, for any $u \in W$, $\sigma(t,u) \in W$ for all $t \geqslant 0$. The following lemma is a variant of [14, Lemma 2.4].

Lemma 5.1. *Assume I satisfies the (PS)-condition, and $c \in \mathbf{R}$ is fixed. Assume $W = \partial W \cup \operatorname{int}(W)$ is an invariant subset w.r.t. σ such that $\sigma(t, \partial W) \subset \operatorname{int}(W)$ for $t > 0$. Define $K_c^1 := K_c \cap W$, $K_c^2 := K_c \cap (X \setminus W)$, where $K_c := \{u \in X : I'(u) = 0, I(u) = c\}$. Let $\delta > 0$ be such that $(K_c^1)_\delta \subset W$ where $\left(K_c^1\right)_\delta = \{u \in X : \operatorname{dist}(u, K_c^1) < \delta\}$. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists $\eta \in C([0,1] \times X, X)$ satisfying*:

(i) $\eta(t,u) = u$ *for* $t = 0$ *or* $u \notin I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \setminus \left(K_c^2\right)_\delta$.
(ii) $\eta\left(1, I^{c+\varepsilon} \cup W \setminus \left(K_c^2\right)_{3\delta}\right) \subset I^{c-\varepsilon} \cup W$ *and* $\eta(1, I^{c+\varepsilon} \cup W) \subset I^{c-\varepsilon} \cup W$ *if* $K_c^2 = \emptyset$.
(iii) $\eta(t,\cdot)$ *is a homeomorphism of* X *for* $t \in [0,1]$.
(iv) $\|\eta(t,u) - u\| \leqslant \delta$, *for any* $(t,u) \in [0,1] \times X$.
(v) $I(\eta(t,\cdot))$ *is non-increasing.*
(vi) $\eta(t, W) \subset W$ *for any* $t \in [0,1]$.
(vii) $\eta(t,\cdot)$ *is odd if* I *is even and if* W *is symmetric with respect to* 0.

Proof. Due to the (PS) condition, we may choose $\varepsilon_0 > 0$ such that

$$\frac{\|I'(u)\|^2}{1 + \|I'(u)\|} \geqslant \frac{8\varepsilon_0}{\delta},$$

for any $u \in I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \setminus (K_c)_\delta$. Set $X_1 := I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \setminus (K_c^2)_\delta$. For any fixed $0 < \varepsilon < \varepsilon_0$, define $X_2 = I^{-1}([c-\varepsilon, c+\varepsilon]) \setminus (K_c^2)_{2\delta}$ and

$$\psi(u) = \frac{\operatorname{dist}(u, X \setminus X_1)}{\operatorname{dist}(u, X \setminus X_1) + \operatorname{dist}(u, X_2)}.$$

Consider

$$\frac{d}{dt}\xi(t,u) = -\frac{\psi(\xi(t,u))V(\xi(t,u))}{1 + \|V(\xi(t,u))\|}, \quad \xi(0,u) = u.$$

Then $\xi(t,u)$ is well-defined and continuous on $\mathbf{R} \times X$ and we claim that $\eta(t,u) := \xi(\delta t, u)$ has all the properties in the lemma. (i), (iii), (v), (vi) and (vii) are easily checked. For (iv) we note that

$$\|\eta(t,u) - u\| = \left\| \int_0^1 \eta'(\tau,u)d\tau \right\| = \left\| \int_0^\delta \xi'(\tau,u)d\tau \right\| \leqslant \delta.$$

To show (ii), we suppose by contradiction that $\eta(1,u) \notin I^{c-\varepsilon} \cup W$ for some $u \in I^{c+\varepsilon} \cup W \setminus \left(K_c^2\right)_{3\delta}$. Then $\eta(t,u) \notin I^{c-\varepsilon} \cup W$ for all $0 \leqslant t \leqslant 1$. We see that $\eta(t,u) \notin$

$(K_c^1)_\delta$ for $0\leqslant t\leqslant 1$ since $(K_c^1)_\delta\subset W$, and $\eta(t,u)\notin (K_c^2)_{2\delta}$ for $0\leqslant t\leqslant 1$ due to (iv) and $u\notin (K_c^2)_{3\delta}$. Thus, for $0\leqslant t\leqslant 1$, $\eta(t,u)\in I^{-1}([c-\varepsilon,c+\varepsilon])\setminus\big((K_c^2)_{2\delta}\cup(K_c)_\delta\big)$, which implies, for $0\leqslant t\leqslant 1$,

$$\psi(\eta(t,u))=1 \quad \text{and} \quad \frac{\|I'(\eta(\tau,u))\|^2}{1+\|I'(\eta(\tau,u))\|}\geqslant\frac{8\varepsilon_0}{\delta}.$$

Then

$$I(\eta(1,u))=I(u)+\int_0^1\frac{\mathrm{d}}{\mathrm{d}\tau}I(\eta(\tau,u))d\tau\leqslant c+\varepsilon-\int_0^1\frac{\delta\|I'(\eta(\tau,u))\|^2}{4(1+\|I'(\eta(\tau,u))\|)}\,d\tau<c-\varepsilon,$$

a contradiction. $\square$

We shall also need the notion of genus (c.f. [19,21]). Set

$$\Sigma_\lambda:=\{A\subset H_\lambda\backslash\{0\}:A \text{ is closed and } A=-A\}$$

and let $\gamma(A)$ denote the genus of A, which is defined as the least integer n such that there exists an odd continuous map $\sigma:A\to S^{n-1}$. We refer to [21] for the following properties of genus.

Proposition 5.2. *Let $A,B\in\Sigma_\lambda$, and $h\in C(H_\lambda,H_\lambda)$ be an odd map. Then*

(i) $A\subset B\Rightarrow\gamma(A)\leqslant\gamma(B)$.
(ii) $\gamma(A\cup B)\leqslant\gamma(A)+\gamma(B)$.
(iii) $\gamma(A)\leqslant\gamma(h(A))$.
(iv) *If A is compact, there exists an $N\in\Sigma_\lambda$ such that $A\subset\mathrm{int}(N)\subset N$ and $\gamma(A)=\gamma(N)$.*
(v) *If F is a linear subspace of H_λ with $\dim F=n$, $A\subset F$ is bounded, open and symmetric, and $0\in A$, then $\gamma(\partial_F A)=n$.*

Recall in Section 4, $D_\lambda^\pm=\{u\in H_\lambda \mid \pm u\geqslant e_1^\lambda\}$ with $e_1^\lambda>0$ the first eigenfunction satisfying $\max e_1^\lambda<s_0$ and s_0 given in (4.4). All positive (negative, resp.) solutions of problem (1.1) are contained in D_λ^+ (D_λ^-, resp.) and $(D_\lambda^\pm)_\varepsilon$ are invariant sets if $\varepsilon\leqslant\varepsilon_0$. We need the following result.

Lemma 5.3. *For any $\rho>0$, let $\mathcal{B}_\rho:=\{u\in H_\lambda:\|u\|_\lambda\leqslant\rho\}$. Then*

$$\mathrm{dist}_\lambda\left(\partial\mathcal{B}_\rho\cap(E_1(\lambda))^\perp,D_\lambda^+\cup D_\lambda^-\right)>0.$$

Proof. Assume for the contrary, there exists $(u_n)\in D_\lambda^+$, $(v_n)\in\partial\mathcal{B}_\rho\cap(E_1(\lambda))^\perp$, such that $\mathrm{dist}_\lambda(u_n,v_n)\to 0$. Then $(u_n,e_1^\lambda)_\lambda=(u_n-v_n,e_1^\lambda)_\lambda+(v_n,e_1^\lambda)_\lambda\to 0$ as $n\to\infty$.

But since $u_n \geqslant e_1^\lambda$, we have

$$(u_n, e_1^\lambda)_\lambda = \mu_1^\lambda (u_n, e_1^\lambda)_{L^2} \geqslant \mu_1^\lambda \int_{\mathbf{R}^N} (e_1^\lambda)^2 \, dx \neq 0,$$

a contradiction. □

Remark 5.4. We note that the above result is not true when $D_\lambda^\pm$ is replaced by $P_\lambda^\pm$. That is, for any $\rho > 0$, it can be proved that $\mathrm{dist}_\lambda \left(\partial \mathcal{B}_\rho \cap (E_1(\lambda))^\perp, P_\lambda^+ \cup P_\lambda^- \right) = 0$.

5.2. The proof of Theorem 1.1—the case $d_m > d_n$

We consider the case $d_m > d_n$ first. In this case we have $m \geqslant 1$ since

$$v_{n+1} \leqslant v_m < \liminf_{|s| \to 0} \frac{f(x,s)}{s} \leqslant \limsup_{|s| \to 0} \frac{f(x,s)}{s} < v_{m+1}.$$

We state the result more precisely here.

Theorem 5.5. *Assume* (V_1)–(V_3) *and* (f_0)–(f_5). *Then there exists a* $\Lambda > 0$ *such that for all* $\lambda \geqslant \Lambda$, Eq. (1.1) *has at least* $d_m - d_n$ ($d_m - d_n - 1$, *resp.*) *pairs of nodal solutions having negative critical values provided* $n \geqslant 1$ ($n = 0$, *resp.*).

The following two lemmas are standard and their special cases were proved in [24,25].

Lemma 5.6. *Assume* (V_1)–(V_2), (f_0), (f_2), *and* (f_4) *with* $m \neq 0$. *Then there exist a* $\rho > 0$ *such that for all* $\lambda \geqslant 0$,

$$\sup_{E_m \cap \partial \mathcal{B}_\rho} I_\lambda < 0.$$

Proof. By (f_2) and (f_4), there exist $\delta > 0$, $C_1 > 0$, and $p \in (2, 2^*)$ such that for all x and u, $F(x,u) \geqslant (v_m + \delta) u^2/2 - C_1 |u|^p$. Here $2^* = 2N/(N-2)$ for $N > 2$ and $2^* = +\infty$ for $N = 2$. For $u \in E_m$, the inequality $\|u\|_\lambda^2 \leqslant v_m \|u\|_{L^2}^2$ and the Sobolev inequality imply

$$\begin{aligned} I_\lambda(u) &\leqslant \frac{1}{2}\|u\|_\lambda^2 - (v_m + \delta) \int_{\mathbf{R}^N} u^2 \, dx - C_1 \int_{\mathbf{R}^N} |u|^p \, dx \\ &\leqslant -\frac{\delta}{2v_m} \|u\|_\lambda^2 + C_2 \|u\|_\lambda^p, \end{aligned}$$

which gives the result. □

Lemma 5.7. *Assume* (V_1)–(V_3), (f_0), (f_3), *and* (f_4). *Then there exists a* $\Lambda > 0$ *such that for all* $\lambda \geqslant \Lambda$, $\dim E_n(\lambda) = d_n$ *and*

$$\inf_{(E_n(\lambda))^\perp} I_\lambda > -\infty.$$

Proof. By (f_3), there exist $\delta > 0$ and $T > 0$ such that for $|u| \geqslant T$, $F(x,u) \leqslant (v_{n+1} - 2\delta)u^2/2$. Choose $\Lambda > 0$ such that for all $\lambda \geqslant \Lambda$, $\mu_{d_n+1} \geqslant v_{n+1} - \delta$ by Lemma 2.2. Let $u \in (E_n(\lambda))^\perp$ with $\lambda \geqslant \Lambda$ and denote $\mathcal{Z} := \{x : |u(x)| \geqslant T\}$. For any $R > 0$, the inequality $\|u\|_\lambda^2 \geqslant \mu_{d_n+1}^\lambda \|u\|_{L^2}^2$ implies

$$\begin{aligned} I_\lambda(u) &\geqslant \frac{1}{2}\|u\|_\lambda^2 - \int_{B_R \cap \mathcal{Z}^c} F(x,u)\,dx - \frac{1}{2}c_0 \int_{B_R^c \cap \mathcal{Z}^c} u^2\,dx - \frac{\mu_{d_n+1}^\lambda - \delta}{2} \int_{\mathcal{Z}} u^2\,dx \\ &\geqslant \frac{\delta}{2\mu_{d_n+1}^\lambda}\|u\|_\lambda^2 - C_1(R) - \frac{1}{2}c_0\|u\|_{L^2(B_R^c)}^2. \end{aligned}$$

Then using Lemma 2.1 gives the result. $\square$

Proof of Theorem 5.5. Throughout the proof, we fix a λ sufficiently large such that I_λ satisfies the (PS)-condition (Lemma 2.3) and all other relevant results hold. By Theorem 4.2 we may choose an $\varepsilon > 0$ small such that $(D_\lambda^+)_\varepsilon \cap (D_\lambda^-)_\varepsilon = \emptyset$ and $W := (D_\lambda^+)_\varepsilon \cup (D_\lambda^-)_\varepsilon$ is an invariant set of the pseudo gradient flow. Recall that $\mathrm{int}(W)$ contains all positive and negative solutions. For $j = 2, \ldots, d_m$, define $\Gamma_j := \{A \in \Sigma_\lambda : \gamma(A) \geqslant j\}$ and

$$c_j := \inf_{A \in \Gamma_j} \sup_{A \cap S} I_\lambda(u),$$

where $S := H_\lambda \backslash W$. For any $A \in \Sigma_\lambda$ with $\gamma(A) \geqslant 2$, we have $A \cap S \neq \emptyset$ (since, if $A \cap S = \emptyset$, $A \subset W$, but $\gamma(W) = 1$, a contradiction). Thus c_j for $j = 2, \ldots, d_m$ can be defined.

First we assume $d_n = 0$ and we consider c_j for $j = 2, \ldots, d_m$. As a consequence of Lemmas 5.6 and 5.7, $-\infty < c_2 \leqslant c_3 \leqslant \cdots \leqslant c_{d_m} < 0$. We claim that if $c := c_j = c_{j+1} = \cdots = c_{j+k}$ for some $2 \leqslant j \leqslant j+k \leqslant d_m$ with $k \geqslant 0$ then $\gamma(K_c \cap S) \geqslant k+1$. Since $K_c^2 = K_c \cap S$ is compact, there exists a closed neighborhood N in H_λ with $K_c^2 \subset \mathrm{int}(N) \subset N$ such that $\gamma(N) = \gamma(K_c^2)$. Without loss of generality, we may assume $N = \overline{(K_c^2)_{3\delta}}$ with $\delta > 0$ satisfying $(K_c^1)_\delta \subset W$. Then there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists an $\eta \in C([0,1] \times H_\lambda, H_\lambda)$ satisfying (i)–(vii) of Lemma 5.1. Choose $A \in \Gamma_{j+k}$ such that

$$\sup_{A \cap S} I_\lambda \leqslant c + \varepsilon,$$

Then $\eta\left(1, A \setminus \left(K_c^2\right)_{3\delta}\right) \subset I_\lambda^{c-\varepsilon} \cup W$ and we claim

$$\gamma\left(\eta\left(1, A\setminus\left(K_c^2\right)_{3\delta}\right)\right) \leqslant j-1.$$

Otherwise, $\gamma\left(\eta\left(1, A\setminus\left(K_c^2\right)_{3\delta}\right)\right) \geqslant j$ and

$$c = c_j \leqslant \sup_{\eta(1,A\setminus(K_c^2)_{3\delta})\cap S} I_\lambda \leqslant \sup_{(I_\lambda^{c-\varepsilon}\cup W)\cap S} I_\lambda \leqslant \sup_{I_\lambda^{c-\varepsilon}} I_\lambda \leqslant c-\varepsilon,$$

a contradiction. Now,

$$\begin{aligned} j+k \leqslant \gamma(A) &\leqslant \gamma(A\setminus \mathrm{int}(N)) + \gamma(N) \\ &\leqslant \gamma(\eta(1, A\setminus(K_c^2)_{3\delta})) + \gamma(K_c^2) \\ &\leqslant j-1+\gamma(K_c^2), \end{aligned}$$

which implies $\gamma(K_c^2) \geqslant k+1$. Since all the solutions in K_c^2 are nodal solutions, the result is true in the case $d_n = 0$.

If $d_n \geqslant 1$, we may consider all c_j for $j = d_n+1, \ldots, d_m$. By Lemmas 5.6 and 5.7, $-\infty < c_{d_n+1} \leqslant \cdots \leqslant c_{d_m} < 0$. The same argument as above works in this case. □

5.3. The proof of Theorem 1.1—the case $d_n > d_m$

In this case we have $n \geqslant 1$ and we prove the following result here which gives the other part of Theorem 1.1.

Theorem 5.8. *Assume (V_1)–(V_3) and (f_0)–(f_5). Then there exists a $\Lambda > 0$ such that for all $\lambda \geqslant \Lambda$, Eq. (1.1) has at least $d_n - d_m$ ($d_n - d_m - 1$, resp.) pairs of nodal solutions having positive critical values provided $m \geqslant 1$ ($m = 0$, resp.).*

We need the following two lemmas which are generalizations of similar results in [24].

Lemma 5.9. *Assume (V_1)–(V_2), (f_0), (f_3), and (f_4) with $n \neq 0$. Then there exists a $R > 0$ such that for all $\lambda \geqslant 0$ and $\mathcal{B}_R^c := H_\lambda \setminus \mathcal{B}_R$,*

$$\sup_{E_n \cap \mathcal{B}_R^c} I_\lambda < 0.$$

Proof. By (f_3), there exist $\delta > 0$ and $T > 0$ such that for $|u| \geqslant T$, $F(x,u) \geqslant (v_n + \delta)u^2/2$. Let $u \in E_n$ and denote $\mathcal{Z} := \{x : |u(x)| \geqslant T\}$. For any $R > 0$,

$$\begin{aligned} I_\lambda(u) &\leqslant \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{2}(c_0 + v_n + \delta)\int_{\mathcal{Z}^c} u^2\,dx - \frac{v_n+\delta}{2}\int_{\mathbf{R}^N} u^2\,dx \\ &\leqslant -\frac{\delta}{2v_n}\|u\|_\lambda^2 + C_1(R) + \frac{1}{2}(c_0 + v_n + \delta)\|u\|^2_{L^2(B_R^c)}, \end{aligned}$$

this together with Lemma 2.1 leads to the result. □

Lemma 5.10. *Assume* (V_1)–(V_2), (f_0), (f_2), *and* (f_4). *Then there exist a* $\Lambda > 0$ *and* $\rho > 0$ *such that for all* $\lambda \geqslant \Lambda$, $\dim E_m(\lambda) = d_m$ *and*

$$\inf_{(E_m(\lambda))^\perp \cap \partial\mathcal{B}_\rho} I_\lambda > 0.$$

Proof. There exist $\delta > 0$ and $C_1 > 0$ such that $F(x,u) \leqslant (v_{m+1} - 2\delta)u^2/2 + C_1|u|^p$ for some $p \in (2, 2^*)$. Choose $\Lambda > 0$ such that for $\lambda > \Lambda$, $\mu^\lambda_{d_m+1} > v_{m+1} - \delta$. Then for $\lambda \geqslant \Lambda$ and $u \in (E_m(\lambda))^\perp$,

$$\begin{aligned} I_\lambda(u) &\geqslant \frac{1}{2}\|u\|_\lambda^2 - \frac{\mu^\lambda_{d_m+1} - \delta}{2}\int_{\mathbf{R}^N} u^2 - C_1\|u\|^p_{L^p} \\ &\geqslant \frac{\delta}{2\mu^\lambda_{d_m+1}}\|u\|_\lambda^2 - C_2\|u\|_\lambda^p, \end{aligned}$$

which gives the result. □

Proof of Theorem 5.8. We need to distinguish two subcases here: (i) $d_m \geqslant 1$ and (ii) $d_m = 0$.

Let us consider (i) first. Again, by Theorem 4.2 we may choose an ε small enough such that $W := (D_\lambda^+)_\varepsilon \cup (D_\lambda^-)_\varepsilon$ is an invariant set of the gradient flow and all positive and negative solutions are contained in $\mathrm{int}(W)$. Set $S = H_\lambda \setminus W$. We have to use a different family of sets for the minimax procedure here. We essentially follow [19]. Define,

$$G := \{h \in C(\mathcal{B}_R \cap E_n, H_\lambda) : h \text{ is odd and } h = id \text{ on } \partial\mathcal{B}_R \cap E_n\},$$

where $R > 0$ is given by Lemma 5.9. Note that $G \neq \emptyset$, since $id \in G$. Set

$$\tilde{\Gamma}_j := \left\{h\left(\overline{\mathcal{B}_R \cap E_n \setminus Y}\right) : h \in G, Y \in \Sigma_\lambda \text{ and } \gamma(Y) \leqslant d_n - j\right\}$$

for $j \in \{2, \ldots, d_n\}$. From [19], $\tilde{\Gamma}_j$ possess the following properties:

(1°) $\tilde{\Gamma}_j \neq \emptyset$ for all $j \in \{2, \ldots, d_n\}$.
(2°) $\tilde{\Gamma}_{j+1} \subset \tilde{\Gamma}_j$ for $j \in \{2, \ldots, d_n - 1\}$.
(3°) If $\sigma \in C(H_\lambda, H_\lambda)$ is odd and $\sigma = id$ on $\partial\mathcal{B}_R \cap E_n$, then $\sigma : \tilde{\Gamma}_j \to \tilde{\Gamma}_j$ for all $j \in \{2, \ldots, d_n\}$ (i.e. $\sigma(A) \in \tilde{\Gamma}_j$ if $A \in \tilde{\Gamma}_j$).
(4°) If $A \in \tilde{\Gamma}_j$, $Z \in \Sigma_\lambda$, $\gamma(Z) \leqslant s < j$ and $j - s \geqslant 2$, then $\overline{A\backslash Z} \in \tilde{\Gamma}_{j-s}$.

Now, for $j = d_m + 1, \ldots, d_n$, we define

$$\tilde{c}_j := \inf_{A \in \tilde{\Gamma}_j} \sup_{A \cap S} I_\lambda.$$

From [19, Proposition 9.23] for $A \in \tilde{\Gamma}_j$ with $j \geqslant d_m + 1$,

$$A \cap \partial\mathcal{B}_\rho \cap (E_m(\lambda))^\perp \neq \emptyset.$$

Since $d_m \geqslant 1$, by Lemma 5.3, $\partial\mathcal{B}_\rho \cap (E_m(\lambda))^\perp \subset S$. Thus, for $j \geqslant d_m + 1$ and $A \in \tilde{\Gamma}_j$, $A \cap S \neq \emptyset$, and from Lemma 5.10 we conclude that

$$\tilde{c}_j \geqslant \inf_{\partial\mathcal{B}_\rho \cap (E_m(\lambda))^\perp} I_\lambda \geqslant \alpha > 0.$$

Then from the definition of $\tilde{c}_j$ and (2°) we have $0 < \alpha \leqslant \tilde{c}_{d_m+1} \leqslant \cdots \leqslant \tilde{c}_{d_n} < \infty$. We claim that if $\tilde{c} := \tilde{c}_j = \cdots = \tilde{c}_{j+k}$ for some $d_m + 1 \leqslant j \leqslant j + k \leqslant d_n$ with $k \geqslant 0$ then $\gamma(K_{\tilde{c}} \cap S) \geqslant k + 1$. This also shows that each $\tilde{c}_j$ is a critical value. Since $0 \notin K_{\tilde{c}}$, and $K^2_{\tilde{c}} = K_{\tilde{c}} \cap S$ is compact, we may choose N such that $K^2_{\tilde{c}} \subset \mathrm{int}(N) \subset N$ and $\gamma(K^2_{\tilde{c}}) = \gamma(N)$. If $\gamma(K^2_{\tilde{c}}) \leqslant k$, we have $\gamma(N) \leqslant k$. By Lemma 5.1 there exist $\varepsilon > 0$ and $\eta \in C([0, 1] \times H_\lambda, H_\lambda)$ such that $\eta(1, \cdot)$ is odd, $\eta(1, u) = u$ for $u \in I^{\tilde{c}-2\varepsilon}$, and

$$\eta\left(1, I_\lambda^{\tilde{c}+\varepsilon} \cup W\backslash N\right) \subset I_\lambda^{\tilde{c}-\varepsilon} \cup W.$$

We may assume $\tilde{c} - 2\varepsilon > 0$. Choose $A \in \tilde{\Gamma}_{j+k}$ such that

$$\sup_{A \cap S} I_\lambda \leqslant \tilde{c} + \varepsilon.$$

Then by (4°) above $\overline{A\backslash N} \in \tilde{\Gamma}_j$. As a consequence of Lemma 5.9, $\eta(1, u) = u$ for $u \in \partial\mathcal{B}_R \cap E_n$, and we have $\eta(1, \overline{A\backslash N}) \in \tilde{\Gamma}_j$ by (3°). Then

$$\tilde{c} \leqslant \sup_{\eta(1,\overline{A\backslash N}) \cap S} I_\lambda \leqslant \sup_{\left(I_\lambda^{\tilde{c}-\varepsilon} \cup W\right) \cap S} I_\lambda \leqslant \tilde{c} - \varepsilon,$$

a contradiction. Therefore $\gamma(K_{\tilde{c}} \cap S) \geqslant k+1$ and I_λ has at least $d_n - d_m$ pairs of nodal critical points.

Next we consider case (ii): $d_m = 0$. We have to use a different invariant set since 0 is a local minimum. From Theorem 4.1, we may choose an $\varepsilon > 0$ sufficiently small such that $(P_\lambda^{\pm})_\varepsilon$ are invariant sets. Set $W := (P_\lambda^+)_\varepsilon \cup (P_\lambda^-)_\varepsilon$ and $S^* := H_\lambda \backslash W$. We define

$$c_j^* := \inf_{A \in \tilde{\Gamma}_j} \sup_{A \cap S^*} I_\lambda,$$

for $j = 2, \ldots, d_n$. We need to show that for any $A \in \tilde{\Gamma}_j$ and $j = 2, \ldots, d_n$, $A \cap S^* \neq \emptyset$ so that c_j^* are well defined, and $c_2^* \geqslant \alpha > 0$.

Consider the attracting domain of 0 in H_λ:

$$Q := \{u \in H_\lambda : \sigma(t, u) \to 0, \text{ as } t \to \infty\}.$$

Note that ∂Q is an invariant set. We claim that for $A \in \tilde{\Gamma}_j$ with $j = 2, \ldots, d_n$, it holds

$$A \cap S^* \cap \partial Q \neq \emptyset. \tag{5.2}$$

This proves both $A \cap S^* \neq \emptyset$ and $c_2^* \geqslant \alpha > 0$, since $\partial \mathcal{B}_\rho \subset Q$ and $\inf_{\partial Q} I_\lambda \geqslant \inf_{\partial \mathcal{B}_\rho} I_\lambda \geqslant \alpha > 0$ by Lemma 5.10. To prove (5.2), let $A = h(\overline{\mathcal{B}_R \cap E_n \backslash Y})$ with $\gamma(Y) \leqslant d_n - j$ and $j \geqslant 2$. Define

$$\mathcal{O} := \{u \in \mathcal{B}_R \cap E_n : h(u) \in Q\}.$$

Then $\mathcal{O}$ is a bounded open set with $0 \in \mathcal{O}$ and $\overline{\mathcal{O}} \subset \mathcal{B}_R \cap E_n$. Thus, from the Borsuk–Ulam theorem $\gamma(\partial\mathcal{O}) = d_n$ and by the continuity of h, $h(\partial\mathcal{O}) \subset \partial Q$. It follows that $\gamma(\overline{\partial\mathcal{O} \backslash Y}) \geqslant j$, $h(\overline{\partial\mathcal{O} \backslash Y}) \subset A \cap \partial Q$ and therefore $\gamma(A \cap \partial Q) \geqslant j$. Since $\gamma(W \cap \partial Q) = 1$, which follows from $(P_\lambda^+)_\varepsilon \cap (P_\lambda^-)_\varepsilon \cap \partial Q = \emptyset$, we conclude that

$$\gamma(A \cap S^* \cap \partial Q) \geqslant \gamma(A \cap \partial Q) - \gamma(W \cap \partial Q) \geqslant 1,$$

which proves (5.2). Thus c_j^* are well defined for $j = 2, \ldots, d_n$ and $0 < \alpha \leqslant c_2^* \leqslant c_3^* \leqslant \cdots \leqslant c_{d_n}^* < \infty$. Proceeding as for the case $d_m > 0$ we have: if $c_j^* = c_{j+1}^* = \cdots = c_{j+k}^*$ for $2 \leqslant j \leqslant j+k \leqslant d_n$ with $k \geqslant 0$ then $\gamma(K_{c_j^*} \cap S^*) \geqslant k+1$. Therefore, I_λ has at least $d_n - d_m - 1$ pairs of nodal critical points. The proof is similar and we omit it. Then the proof for the case (ii) is finished, and therefore the proof of the main Theorem 1.1 is complete. $\square$

6. Related results

Our method allows us to work on several other problems in the entire space $\mathbf{R}^N$ and we mention some results here with sketch of the proofs.

First we observe that additional information on the signed solutions will produce stronger existence results. In case $d_n > d_m \geqslant 1$, we have $d_n - d_m$ pairs of nodal solutions. The following result gives additional nodal solutions to those in Theorem 1.1 in the presence of a pair of signed solutions.

Theorem 6.1. *Assume* (V_1)–(V_3) *and* (f_0)–(f_5). *Assume that there exists a positive solution* w (*so* $-w$ *is a negative solution*). *If* $d_n > d_m \geqslant 2$, *Eq.* (1.1) *has at least* $d_n - 1$ *pairs of nodal solutions with* $d_n - d_m$ *pairs having positive critical values and* $d_m - 1$ *pairs having negative critical values.*

Sketch of the proof. Using Theorem 1.1 we get $d_n - d_m$ pairs of nodal solutions having positive critical values. Due to the presence of a pair of signed solutions we can also get $d_m - 1$ pairs of nodal solutions having negative critical values. This is done by modifying $f(x, u)$ to a new function $\hat{f}(x, u)$ as in Section 3 and consider Eq. (3.1) in Section 3. It is easy to check the conditions of Theorem 1.1 are satisfied with $d_n = 0$ in this case, so applying Theorem 1.1 we get $d_m - 1$ pairs of nodal solutions with negative critical values. $\square$

Next, we consider the following problem.

$$-\Delta u + V(x)u = f(x, u), \quad \text{in } \mathbf{R}^N, \tag{6.1}$$

which satisfy $u(x) \to 0$ as $|x| \to \infty$. The potential function V satisfies

$(V_{1'})$ $V \in C(\mathbf{R}^N, \mathbf{R})$ satisfies $\inf V \geqslant V_0 > 0$.
$(V_{2'})$ $\lim_{|x|\to\infty} V(x) = +\infty$.

Under these conditions the linear operator is compact and has discrete spectrum only, i.e., $\sigma(-\Delta + V)$ is given by $0 < v_1 < v_2 < \cdots$ with the dimension of each eigenvalue $\dim(v_k) < \infty$. Thus this is somewhat simpler case than the one we have considered. We state the conditions and a similar result and leave the details to the readers.

We may assume the more general condition than $(V_{2'})$

$(V_{2''})$ There exists $r_0 > 0$ such that for any $M > 0$

$$\lim_{|y|\to\infty} m\left(\{x \in \mathbf{R}^N : |x - y| \leqslant r_0\} \cap \{x \in \mathbf{R}^N : V(x) \leqslant M\}\right) = 0.$$

Again, set

$$d_k := \sum_{i=1}^{k} \dim(v_i) \quad \text{and} \quad d_0 := 0.$$

Theorem 6.2. *Assume* $(V_{1'})$, $(V_{2''})$, *and* (f_0)–(f_5). *Then Eq.* (6.1) *has at least* $|d_m - d_n|$ *pairs of nodal solutions provided* $\min\{m, n\} \geqslant 1$, *and at least* $|d_m - d_n| - 1$ *pairs of nodal solutions if* $\min\{m, n\} = 0$.

There is a version of Theorem 6.1 for Eq. (6.1) too.

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