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ON THE EXISTENCE OF SIGN CHANGING SOLUTIONS FOR SEMILINEAR DIRICHLET PROBLEMS

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction

We consider the semilinear Dirichlet problem

$$\text{(D)} \qquad -\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and $f : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ with $f(0) = 0$. Thus $u_0 \equiv 0$ is a trivial solution of (D) and we are interested in finding and studying nontrivial solutions. One way of obtaining these is to compare the behavior of f near the origin and near infinity. We shall always assume that f grows subcritically at infinity so that variational methods can be applied and the associated functional satisfies the Palais–Smale condition.

Suppose $f'(0) < \lambda_1$ where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ are the eigenvalues (counted with multiplicities) of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions. If f grows superlinearly at infinity then the mountain pass theorem of Ambrosetti and Rabinowitz [AR], [R] together with the maximum principle guarantees the existence of a positive solution u_+ and a negative solution u_- of (D). Using linking or Morse type arguments Wang [Wa] obtained a third nontrivial solution u_1. In this paper we shall refine Wang's result and obtain more information on u_1 and on other solutions whose existence is proved via Morse theory. Let us illustrate this with the following two theorems. More general results will be stated and proved later.

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Theorem 1. *Suppose $f'(0) < \lambda_2$ and f grows superlinearly but subcritically at infinity. Then there exists a solution u_1 of* (D) *which changes sign. If u_2 is a second nontrivial solution then $u_2 > u_1$* (*respectively, $u_2 < u_1$*) *implies that u_2 is positive* (*respectively, negative*)*. If $f'(0) < \lambda_1$ then there exist a positive solution u_+ of* (D) *and a negative solution u_- such that $u_1 - u_+$ and $u_1 - u_-$ both change sign.*

If $f'(0) < \lambda_1$ the existence of three solutions is well known. Observe that the solution u_1 exists even in the resonant case $f'(0) = \lambda_1$ without any further condition on the behavior of f near 0. This seems to be new. Our main new observation, however, is that the Morse type arguments which yield the existence of u_1 can be used in combination with the maximum principle to prove that u_1 changes sign and to obtain information on the relation of other solutions to u_1 and $u_0 \equiv 0$. We only know of the paper [CCN] by Castro *et al.* where the existence of a sign changing solution is proved using much stronger hypotheses on f however.

Theorem 2. *Suppose $f'(t) \to \omega \in \mathbb{R}$ as $|t| \to \infty$.*

(a) *If neither ω nor $f'(0)$ are eigenvalues of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions and if there exists $k \geq 2$ such that λ_k lies between ω and $f'(0)$ then* (D) *has a sign changing solution.*

(b) *If $\omega < \lambda_2 < f'(0)$ then* (D) *has a sign changing solution u_1 even if $\omega = \lambda_1$ and $f'(0) = \lambda_k$ for some $k \geq 3$. Moreover, any positive solution is larger than u_1 and any negative solution is smaller than u_1.*

(c) *If $\omega < \lambda_1 < \lambda_2 < f'(0)$ then* (D) *has three nontrivial solutions $u_+ > 0$, $u_- < 0$ and u_1 such that $u_-(x) < u_1(x) < u_+(x)$ for every $x \in \Omega$. Moreover, any other positive solution is larger than u_+ and any other negative solution is smaller than u_-.*

The existence of the solutions is well known if $f'(0)$ and ω are not eigenvalues of $-\Delta$. In that case and assuming $\omega < \lambda_1$ Hofer [H] proved even the existence of four nontrivial solutions u_+, u_-, u_1, u_2 of (D) using degree theory. Since we allow $f'(0)$ to be an eigenvalue the degree of the trivial solution may be 0 and all its critical groups (see below) may vanish.

In order to prove results of this type we develop new variational methods for functionals Φ defined on partially ordered Hilbert spaces whose gradient is of the form $\nabla\Phi = \mathrm{Id} - K$ where K is a compact and order preserving nonlinear operator. For the application to (D) we have

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(u) \, dx$$

where $F(u) = \int_0^u f(t) \, dt$ is the primitive of f. That the gradient of Φ (with respect to a properly chosen scalar product on $H_0^1(\Omega)$) is of the above form is a consequence of the maximum principle. We refer the reader to the papers by Amann [A], Chang [Ch1], Hofer [H] or Wysocki [Wy] where this observation is used to prove the existence of positive solutions of (D) under various hypotheses on f. Here we show how the maximum principle can be used to prove that certain solutions change sign. The philosophy of our results is that if the behavior of the energy functional Φ near the origin and near infinity implies the existence of a critical point u_1 whose critical group

$$C_k(\Phi, u_1) := H_k(\Phi^c, \Phi^c - \{u_1\})$$

is not trivial for some $k \geq 2$ then this critical point can be neither positive nor negative. Here H_k denotes singular homology theory with arbitrary coefficients, $c = \Phi(u_1)$ and $\Phi^c = \{u \in H_0^1(\Omega) : \Phi(u) \leq c\}$ is the sublevel set as usual. We emphasize that there may be many positive or negative critical points and they may have nontrivial critical groups in all possible dimensions. Moreover, positive or negative critical points may accumulate at 0. In other words, there will be no assumptions at all on the set of positive or negative solutions.

The paper is organized as follows. In Section 2 we develop some abstract critical point theory for functionals on partially ordered Hilbert spaces which respect the partial order in the sense mentioned above. The results of this section will be proved in Section 3. These two sections form the core of the paper and are of independent interest. Finally, in Section 4 we state and prove generalizations of Theorems 1 and 2.

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2. Critical point theory for functionals on partially ordered Hilbert spaces

Let E be a Hilbert space and $P_E \subset E$ a closed cone, that is, $P_E = \overline{P}_E$ is convex, $\mathbb{R}^+ \cdot P_E \subset P_E$ and $P_E \cap (-P_E) = \{0\}$. As usual, this turns E into a partially ordered space where

$$u \geq v :\Leftrightarrow u - v \in P_E, \quad u > v :\Leftrightarrow u \geq v \text{ and } u \neq v.$$

If $u - v \notin P_E \cup (-P_E)$ then u and v are said to be *noncomparable*. A map $f : E \to E$ is called *order preserving* if

$$u \geq v \Rightarrow f(u) \geq f(v) \quad \text{for all } u, v \in E.$$

Let $X \subset E$ be a Banach space which is densely embedded into E. We set $P := X \cap P_E$ and assume that P has nonempty interior $\mathring{P} \neq \emptyset$. We also assume that there exists an element $e \in \mathring{P}$ such that $\langle u, e\rangle > 0$ for all $u \in P \setminus \{0\}$. Here $\langle -, -\rangle$ denotes the scalar product of E. Thus X is a partially ordered Banach space and we define

$$u \gg v :\Leftrightarrow u - v \in \mathring{P}.$$

The elements of $\mathring{P}$ are called *positive*, those of $-\mathring{P}$ *negative*. A map $f : X \to X$ is said to be *strongly order preserving* if

$$u > v \Rightarrow f(u) \gg f(v) \quad \text{for all } u, v \in X.$$

Next we consider a function $\Phi : E \to \mathbb{R}$ which satisfies the conditions:

(Φ_1) $\Phi \in C^2(E, \mathbb{R})$, $\Phi(0) = 0$, $\Phi'(0) = 0$ and the Palais–Smale condition holds for Φ. Any critical point of Φ lies in X.

(Φ_2) The gradient of Φ is of the form $\nabla\Phi = \mathrm{Id} - K_E$ where $K_E : E \to E$ is a compact (nonlinear) operator. Moreover, $K_E(X) \subset X$ and the restriction $K := K_E|X : X \to X$ is of class $\mathcal{C}^1$ and strongly order preserving.

(Φ_3) Any eigenvector of the (Fréchet) derivative $DK_E(0) \in \mathcal{L}(E)$ lies in X, the largest eigenvalue of $DK_E(0)$ is simple and its eigenspace is spanned by a positive eigenvector.

These assumptions are slightly weaker than the hypotheses ($\Phi\Phi$) of [H]. There it is assumed in addition that K_E and $DK_E(u) \in \mathcal{L}(E)$ are regular in the following sense: There exists a finite sequence $E = E_n \supset E_{n-1} \supset \ldots \supset E_1 \supset E_0 = X$ of Banach spaces E_i such that K and $DK(u)$, $u \in X$, induce continuous operators $E_i \to E_{i-1}$ for $i = 1, \ldots, n$. Moreover, it is assumed that K_E and $DK_E(u)$ are order preserving and $DK(u) \in \mathcal{L}(X)$, $u \in X$, is strongly order preserving. Then (Φ_3) is a consequence of the Krein–Rutman theorem.

Finally, we need an assumption on the behavior of Φ near infinity. We shall distinguish between the cases of Φ coercive, Φ superquadratic, and Φ asymptotically quadratic.

(Φ_4) One of the following holds:

(i) Φ is bounded below.

(ii) For every $u \in E - \{0\}$ we have $\Phi(tu) \to -\infty$ as $t \to \infty$. There exists $a < 0$ such that $\Phi(u) \le a$ implies $\Phi'(u)u < 0$.

(iii) There exists a compact self-adjoint linear operator $A_E \in \mathcal{L}(E)$ such that $\nabla\Phi(u) = u - A_E u + o(\|u\|_E)$ as $\|u\|_E \to \infty$. All eigenvectors of A_E lie in X, the largest eigenvalue is simple and its eigenspace is spanned by a positive eigenvector $v_\infty \in \mathring{P}$ such that $\langle u, v_\infty\rangle > 0$ for every $u \in P \setminus \{0\}$. Moreover, the restriction $A := A_E|_X$ is a bounded linear operator $A \in \mathcal{L}(X)$.

By a result of Kaclovic *et al.* [KLW], Φ must be coercive if (Φ_4)(i) holds and the Palais–Smale condition is satisfied. Case (ii) in (Φ_4) models the superlinear growth of f in the application to (D) whereas case (iii) corresponds to the asymptotically linear growth of f near infinity.

Now we define the *Morse index* $\mu_\infty \in \mathbb{N} \cup \{\infty\}$ and the *nullity* $\nu_\infty \in \mathbb{N}$ of Φ at infinity depending on whether (i), (ii) or (iii) holds in (Φ_4). In case (i) we set $\mu_\infty := 0$ and in case (ii) we set $\mu_\infty := \infty$. In both cases the nullity is trivial, $\nu_\infty := 0$. If (iii) holds then μ_∞ is the dimension of the negative eigenspace of $\mathrm{Id} - A_E$, that is, the number of eigenvalues of A_E which are larger than 1, counted with multiplicities. We do allow that $\mathrm{Id} - A_E$ has a kernel, so the nullity $\nu_\infty := \dim\ker(\mathrm{Id} - A_E)$ of Φ at infinity may be nonzero. Thus even in the case $\mu_\infty = 0$ it is possible that Φ is not bounded below. Similarly, the origin may be a degenerate critical point of Φ with nullity $\nu_0 := \dim\ker\Phi''(0) = \dim\ker(\mathrm{Id} - DK(0))$ possibly nontrivial.

Theorem 2.1. *Suppose the Morse index μ_0 of Φ at 0 is at least 2 and $\mu_\infty + \nu_\infty \le 1$. Then Φ has a critical point $u_1 \in X$ which is not comparable to 0, that is, $u_1 \notin P \cup (-P)$. Moreover, for any critical point u_2 of Φ the following implications hold: $u_2 > 0$ implies $u_2 \gg u_1$, and $u_2 < 0$ implies $u_2 \ll u_1$.*

If Φ is bounded below then it has a positive critical point $u_+ \in X$ and a negative critical point $u_- \in X$, hence $u_+ \gg u_1 \gg u_-$. For any critical point u_2 of Φ the following implications hold: $u_2 > 0$ implies $u_2 \gg u_+$, and $u_2 < 0$ implies $u_2 \ll u_-$.

Now we consider a dual situation to 2.1.

Theorem 2.2. *Suppose $\mu_\infty \ge 2$ and $\mu_0 + \nu_0 \le 1$. Then Φ has a critical point $u_1 \in X$ which is not comparable to 0. Moreover, for any critical point u_2 of Φ the following implications hold: $u_2 < u_1$ implies $u_2 \ll 0$, and $u_2 > u_1$ implies $u_2 \gg 0$.*

If 0 is a possibly degenerate strict local minimum then Φ has a positive critical point $u_+ \in X$ and a negative critical point $u_- \in X$. For any critical point u_2 of Φ the following implications hold: $u_2 < u_+$ implies $u_2 \ll 0$, and $u_2 > u_-$ implies $u_2 \gg 0$. In particular, u_1 is not comparable to u_+ nor u_-.

The positive and negative critical points in 2.1 and 2.2 are of a different nature. In 2.1 they are local minima whereas in 2.2 they are of mountain pass type. Our last result in this section deals with a situation where both Morse indices μ_0 and μ_∞ may be larger than 1. In that case Φ need not have a positive or a negative critical point even if $\mu_0 \ne \mu_\infty$. There still exists a critical point which is not comparable to 0. For simplicity we only deal with the nondegenerate case at 0 and at infinity.

Theorem 2.3. *Suppose* $\mu_0 \neq \mu_\infty$, $\max\{\mu_0, \mu_\infty\} \geq 2$ *and* $\nu_0 = \nu_\infty = 0$. *Then* Φ *has a critical point which is not comparable to* 0.

Remarks 2.4. a) The nondegeneracy assumptions in 2.3 can be avoided. In that case one has to work with the critical groups $C_*(\Phi, 0) = H_*(\Phi^0, \Phi^0 \backslash \{0\})$ of Φ at 0 and the critical groups $C_*(\Phi, \infty) = H_*(E, \Phi^a)$ of Φ at infinity as discussed in [BL]; here $a < 0$ is a strict lower bound for the critical values of Φ. Essentially a critical point of Φ outside of the union of the cones $P \cup (-P)$ exists if $C_k(\Phi, 0) \neq C_k(\Phi, \infty)$ for some $k \geq 2$. In other words, positive or negative critical points can only contribute to the homology in dimensions $k = 0$ or $k = 1$. Although there may clearly exist positive (and negative) critical points with arbitrary Morse indices their homologies have to cancel each other in dimensions 2 or larger. In [BL] one can find a number of computations of $C_*(\Phi, 0)$ and of $C_*(\Phi, \infty)$ in the degenerate case. In particular, we want to mention Theorem 3.9 of [BL] which states that $C_k(\Phi, \infty) = 0$ for $k \notin [\mu_\infty, \mu_\infty + \nu_\infty]$. The corresponding result for $C_k(\Phi, 0)$ is due to Gromoll and Meyer [GM]. The nontriviality of $C_k(\Phi, 0)$ or $C_k(\Phi, \infty)$ for some k follows from local or global linking conditions or the angle conditions in [BL].

b) The existence of the critical points u_+, u_- in 2.1 and 2.2 is well known (cf. [H]). The existence of u_1 in 2.1 and 2.2 seems to be new in the degenerate case when the nullities ν_0 or ν_∞ are not trivial. In that case the critical groups $C_*(\Phi, 0)$ or $C_*(\Phi, \infty)$ may all be trivial. We refer the reader to the book by Chang [Ch2] and the references therein for existence results. The case $\nu_\infty > 0$ has been treated in [BL]. The main new information contained in 2.1 to 2.3 is the localization of u_1 in relation to the origin and to other critical points. We believe that the method for proving the existence of u_1 is also of interest since it yields automatically the additional information.

c) If 0 is a nondegenerate critical point with Morse index at least 2 and if Φ is coercive then a simple argument using degree theory or the Morse inequalities yields the existence of four nontrivial critical points: a positive and a negative local minimum, a mountain pass type solution and a fourth solution (cf. [H], Theorem 6). The nondegeneracy assumptions at the origin and at infinity are essential for this argument.

3. Proof of 2.1 to 2.3

The main ingredient in the proof of the results from Section 2 is the negative gradient flow φ^t of Φ on E, that is,

$$\frac{d}{dt}\varphi^t = -\nabla\Phi \circ \varphi^t, \quad \varphi^0 = \mathrm{id}.$$

Because of (Φ_2) we have $\varphi^t(u) \in X$ for $u \in X$ and φ^t induces a continuous (local) flow on X which we continue to denote by φ^t. The main order related property of

φ^t is that the positive cone P and the negative cone $-P$ are positively invariant. More generally, we have:

Lemma 3.1. *Suppose $\Phi'(u_0) = 0$, so that $u_0 \in X$ by (Φ_1). Then for every $v \in P - \{0\}$ and every $t > 0$ we have $\varphi^t(u_0 \pm v) \in u_0 \pm \mathring{P}$. Consequently, $u_0 \pm P$ and $u_0 \pm \mathring{P}$ are positively invariant.*

Proof. It suffices to show that for $v \in P\backslash\{0\}$ the vector field $-\nabla\Phi$ points at $u_0 + v$ inside the cone $u_0 + \mathring{P}$, that is,

$$u_0 + v - \nabla\Phi(u_0 + v) \in u_0 + \mathring{P} \quad \text{for } v \in P\backslash\{0\}.$$

This follows easily from (Φ_2):

$$u_0 + v - \nabla\Phi(u_0 + v) = K(u_0 + v) \gg K(u_0) = u_0.$$

The same argument applies to $u_0 - v$. □

Proof of 2.1. Since $\mu_0 \ge 2$ the smallest eigenvalue of $\Phi''(0) = \mathrm{Id} - DK_E(0)$ is negative and the associated eigenspace is spanned by a positive eigenvector. Therefore there exists a subset S of the unstable set $W^u(0) \subset X$ of 0 which is homeomorphic to S^1 and intersects $\mathring{P}$ and $-\mathring{P}$:

$$S \subset W^{\mathrm{u}}(0) = \{u \in X : \varphi^t(u) \to 0 \text{ as } t \to -\infty\}$$
$$S \cong S^1, \quad S \cap \mathring{P} \ne \emptyset \ne S \cap (-\mathring{P}).$$

Now $\mu_\infty + \nu_\infty \le 1$ implies that (Φ_4)(i) or (iii) holds. In case (i) set $v_\infty := e \in \mathring{P}$, in case (iii) let $v_\infty \in \mathring{P}$ be the unique positive eigenvector of $\mathrm{Id} - A$ with $\|v_\infty\| = 1$ which spans the one-dimensional eigenspace of $\mathrm{Id} - A_E$ belonging to the largest eigenvalue of A_E. In any case, Φ is bounded below on $X \cap (\mathrm{span}\{v_\infty\})^\perp$ which is a codimension one subspace of X. We choose any $a < \inf \Phi(X \cap (\mathrm{span}\{v_\infty\})^\perp)$. We claim that the ω-limit set of S contains a critical point u_1 outside of $P \cup (-P)$. Arguing indirectly we suppose that there are no critical points in $\omega(S)\backslash(P \cup (-P))$ where

$$\omega(S) = \{u \in X : \text{there exist sequences } t_n \to \infty, u_n \in S \text{ with } \varphi^{t_n}(u_n) \to u \text{ as } n \to \infty\}$$

is the ω-limit set of S. For every $u \in S$ there exists $\tau(u) \ge 0$ such that $\varphi^t(u) \in \Phi^a \cup \mathring{P} \cup (-\mathring{P})$ for all $t \ge \tau(u)$. By continuity of φ^t and compactness of S there exists $\tau \ge 0$ such that $\varphi^\tau(S) \subset \Phi^a \cup \mathring{P} \cup (-\mathring{P})$. This is not possible, however, since by our assumption on $e \in \mathring{P}$ or on v_∞ in (Φ_4) we have

$$\Phi^a \cup \mathring{P} \cup (-\mathring{P}) \subset X\backslash(\mathrm{span}\{v_\infty\})^\perp$$

and $X\backslash(\mathrm{span}\{v_\infty\})^\perp = X_+ \sqcup X_-$ is the topological sum of the two subsets

$$X_\pm = \{u \in X : \pm\langle v_\infty, u\rangle > 0\}.$$

Clearly $S\cap\mathring{P} \neq \emptyset \neq S\cap(-\mathring{P})$ implies $\varphi^\tau(S)\cap X_\pm \neq \emptyset$ by Lemma 3.1 contradicting the fact that S is connected. This shows that there exists a critical point $u_1 \in \omega(S) \cap X\backslash(P \cup (-P))$. Now let u_2 be a critical point of Φ with $u_2 > 0$. Then $u_2 \in \mathring{P}$ by (Φ_2), hence $0 \in u_2 - \mathring{P}$. Lemma 3.1 then implies that $S \subset u_2 - \mathring{P}$ and therefore $u_1 \in \omega(S) \subset u_2 - \mathring{P}$, that is, $u_1 \ll u_2$.

Finally, if Φ is bounded below we choose $v_+ \in S\cap\mathring{P}$ and $v_- \in S\cap(-\mathring{P})$. Then we have $\omega(v_+) \subset \mathring{P}$ and $\omega(v_-) \subset -\mathring{P}$ by Lemma 3.1 and $\omega(v_\pm) \neq \emptyset$ because Φ is bounded below and satisfies the Palais–Smale condition. Consequently, there exist critical points $u_\pm \in \omega(v_\pm)$. The implications $u_2 > 0 \Rightarrow u_2 \gg u_+$ and $u_2 < 0 \Rightarrow u_2 \ll u_-$ for critical points u_2 of Φ follow as above from $u_\pm \in \omega(v_\pm) \subset \omega(S)$. □

Proof of 2.2. Let $v_0 \in \mathring{P}$ be the unique normalized positive eigenvector of $\Phi''(0) = \mathrm{Id} - DK_E(0)$ which spans the one-dimensional eigenspace belonging to the largest eigenvalue of $DK_E(0)$. Since $\mu_0 + \nu_0 \leq 1$, the stable set of 0,

$$W^{\mathrm{s}}(0) = \{u \in X : \varphi^t(u) \to 0 \text{ as } t \to \infty\}$$

contains a subset $S \subset X\backslash(P \cup (-P))$ of the form

$$S = \{u + \alpha(u) : u \in X \cap (\mathrm{span}\{v_0\})^\perp,\ \|u\|_X = \varepsilon\}$$

where $\alpha : U_{2\varepsilon}(0) \cap (\mathrm{span}\{v_0\})^\perp \to \mathrm{span}\{v_0\}$ is continuous. In fact, if $\mu_0 = 1$ and $\nu_0 = 0$ then the graph of α is the local stable manifold of 0, $\mathrm{graph}(\alpha) = W^{\mathrm{s}}(0) \cap U_{2\varepsilon}(0)$. If $\mu_0 = 0$ and $\nu_0 = 1$ then $\ker \Phi''(0) = \mathrm{span}\{v_0\}$, $\Phi''(0)$ is positive definite on $(\mathrm{span}\{v_0\})^\perp$ and the graph of α describes part of the stable set $W^{\mathrm{s}}(0)$ which need not be a manifold any more in this degenerate case. If $\mu_0 = 0$ and $\nu_0 = 0$ we may choose $\alpha \equiv 0$.

Since $\mu_\infty \geq 2$ there exist $R > 0$ and two orthonormal vectors $v_\infty, w_\infty \in X$ such that $v_\infty \in \mathring{P}$ and $\Phi(u) < 0$, $\Phi'(u)u < 0$ for every $u \in \mathrm{span}\{v_\infty, w_\infty\}$ with $\|u\| \geq R$. This is clear in the case (Φ_4)(ii) when $\mu_\infty = \infty$. If on the other hand, (Φ_4)(iii) applies, then the negative eigenspace of $\mathrm{Id} - A$ has dimension $\mu_\infty \geq 2$ and we may choose $v_\infty \in \mathring{P}$ and w_∞ from this negative eigenspace. Then we set

$$T := \{tv_\infty : -R \leq t \leq R\} \cup \{R(v_\infty \cos\theta + w_\infty \sin\theta) : 0 \leq \theta \leq \pi\}$$

and $C := \mathrm{conv}(T)$, the convex hull of T. Clearly T is homeomorphic to S^1 and C to B^2. An easy degree argument shows that (C, T) and S link. By this we mean that for every continuous deformation $h_t : C \to X$ with $h_0(u) = u$ for $u \in C$ and $h_t(T) \cap S = \emptyset$ for $t \in [0, 1]$, we have $h_t(C) \cap S \neq \emptyset$ for $t \in [0, 1]$. From this we deduce that the ω-limit set of C has nonempty intersection with S, so there exists $v \in \omega(C) \cap S$. From the construction of S it follows that $\varphi^t(v) \to 0$ as $t \to \infty$. In addition, $v \in \omega(C)$ implies $\lim_{t\to-\infty} \Phi(\varphi^t(v)) \leq \max \Phi(C)$. As a consequence of the Palais–Smale condition there exists a critical point u_1 in the α-limit set of v. This has the required properties. First of all, it cannot be comparable to 0 because $u_1 \in \alpha(v) \cap (\mathring{P} \cup (-\mathring{P}))$ would imply $v \in \mathring{P} \cup (-\mathring{P})$ by Lemma 3.1. This is not possible since $v \in S \subset X \backslash (P \cup (-P))$. The implications

$$\Phi'(u_2) = 0,\ u_2 < u_1 \Rightarrow u_2 \ll 0$$

and

$$\Phi'(u_2) = 0,\ u_2 > u_1 \Rightarrow u_2 \gg 0$$

follow as in the proof of Theorem 2.1.

It remains to prove the existence of a positive critical value u_+ and a negative critical value u_- if 0 is a strict local minimum. In fact, the existence of u_+ and u_- is a simple mountain pass argument. We leave it to the reader to show that there even exist $v_+ \in \mathring{P}$ and $v_- \in -\mathring{P}$ with $\varphi^t(v_\pm) \to 0$ as $t \to \infty$ and with nonempty α-limit sets $\alpha(v_+) \subset \mathring{P}$ and $\alpha(v_-) \subset -\mathring{P}$. The proof uses similar ideas to the above and is simpler. Now we choose critical points $u_\pm \in \alpha(v_\pm)$. These satisfy the required implications as usual. □

Proof of 2.3. First we observe that the set of critical values of Φ is bounded below by some $a \in \mathbb{R}$. This is obvious if (i) or (ii) of (Φ_4) holds. In case (iii) it follows easily from the Palais–Smale condition and the nondegeneracy hypothesis that $\nu_\infty = 0$. We fix such a strict lower bound $a \in \mathbb{R}$ and compute the critical groups $C_*(\Phi, \infty) = H_*(E, \Phi^a)$ of Φ at infinity.

Proposition 3.2. *Suppose (Φ_1)–(Φ_4) hold and $\nu_\infty = 0$. Then for any $k \in \mathbb{Z}$ we have*

$$H_k(X, X \cap \Phi^a) \cong H_k(E, \Phi^a) \cong \begin{cases} \mathbb{F} & \text{if } k = \mu_\infty, \\ \{0\} & \text{if } k \neq \mu_\infty. \end{cases}$$

Proof. The isomorphism $H_k(X, X \cap \Phi^a) \cong H_k(E, \Phi^a)$ is a simple consequence of a result of Palais [P]. In the case (Φ_4)(i) we have $\mu_\infty = 0$ and $a < \inf \Phi$, so $\Phi^a = \emptyset$ and $H_k(E, \Phi^a) \cong H_k(pt)$ is as required.

In the case (Φ_4)(ii) the set $\Phi^{-1}(a)$ is radially homeomorphic to the unit sphere of E and Φ^a is radially homotopy equivalent to $\Phi^{-1}(a)$. Since this sphere is contractible we obtain $H_k(E, \Phi^a) \cong \{0\}$ for all $k \in \mathbb{Z}$.

Finally, if (Φ_4)(iii) holds the proposition has been proved in [BL], Theorem 3.9. The assumption (A_∞) in [BL] is slightly different from those considered here but the proof applies without changes since $\nu_\infty = 0$. □

If (Φ_4)(iii) applies and $\nabla\Phi - \mathrm{Id} + A_E \in \mathcal{C}^1(E, E)$ is bounded then a proof of 3.2 can be found in [Ch2].

Proposition 3.3. *Suppose* (Φ_1)–(Φ_4) *hold with* $\mu_0,\mu_\infty \ge 1$ *and* $\nu_0,\nu_\infty = 0$. *If all critical points of* Φ *lie in* $P\cup(-P)$ *then*

$$H_k(X,\Phi_X^a) \cong \begin{cases} \mathbb{F} & \text{if } k=\mu_0,\\ 0 & \text{if } k\ne\mu_0, \end{cases}$$

where $\Phi_X^a := X\cap\Phi^a = \{u\in X : \Phi(u)\le a\}$.

Postponing the proof of Proposition 3.3 we first deduce Theorem 2.3. If $\mu_0 = 0$ then Theorem 2.1 applies. Similarly, if $\mu_\infty = 0$ then 2.3 follows from 2.2. Finally, if $\mu_0,\mu_\infty \ge 1$ and $\mu_0 \ne \mu_\infty$ the existence of a critical point of Φ outside of $P\cup(-P)$ is a consequence of 3.2 and 3.3. This proves Theorem 2.3. □

The rest of this section provides a proof of Proposition 3.3.

Lemma 3.4. *Under the hypotheses of Proposition* 3.3 *every* $u\in X$ *satisfies precisely one of the following conditions.*

(i) $u\in P\cup(-P)\cup\Phi^a$.
(ii) $\varphi^t(u)\notin P\cup(-P)$ *for all* $t\ge 0$ *and* $\varphi^t(u)\to 0$ *as* $t\to\infty$.
(iii) *There exists a unique* $T>0$ *such that*

$$\varphi^t(u)\notin P\cup(-P) \quad \text{for } 0\le t<T, \quad \varphi^T(u)\in P\cup(-P) \quad \text{and} \quad \Phi(\varphi^T(u))\ge a.$$

(iv) *There exists a unique* $T>0$ *such that*

$$\Phi(\varphi^T(u)) = a \quad \text{and} \quad \varphi^T(u)\notin P\cup(-P).$$

Proof. The lemma follows from 3.1 and the Palais–Smale condition. □

Since $\nu_0 = 0$ we can apply the Grobman–Hartman theorem to the flow φ^t on X. This yields a local homeomorphism $\chi:(U,0)\to(X,0)$ such that $\varphi^t\circ\chi(u) = \chi\circ e^{-Lt}u$. Here $L := \mathrm{Id} - DK(0)\in\mathcal{L}(X)$, U is a neighborhood of 0 in X and the conjugacy holds for $u, e^{-Lt}u\in U$. We split $X = V\oplus W$ into the positive eigenspace V and the negative eigenspace W of $-L$, hence $\dim V = \mu_0$. We also write $u = v+w\in V+W$ according to this decomposition. For $\varepsilon>0$ with $U_{2\varepsilon}(0)\subset U$ we set

$$\begin{aligned} A &:= \{\chi(v+w) : v\in V,\ w\in W,\ \|v\|_X, \|w\|_X \le\varepsilon\},\\ B &:= \{\chi(v+w) : v\in V,\ w\in W,\ \|v\|_X = \varepsilon,\ \|w\|_X\le\varepsilon\}. \end{aligned}$$

By Lemma 3.4 there exists $T\ge 0$ such that

$$\varphi^T(B)\subset \mathring{P}\cup(-\mathring{P})\cup\Phi^{a-1}.$$

This is clear since the case 3.4(ii) does not apply for $u\in B$ and since

$$\|u-K(u)\|_X \ge \text{const}\cdot\|u-K(u)\|_E = \text{const}\cdot\|\nabla\Phi(u)\|_E$$

is bounded away from 0 uniformly for $u \in (\bigcup_{t\geq 0}\varphi^t(B))\backslash(P\cup(-P)\cup\Phi^{a-1})$. The last statement follows from the Palais–Smale condition. Now we consider the set

$$Z := P \cup (-P) \cup \varphi^T(A) \cup \Phi_X^a.$$

Lemma 3.5. *Under the hypotheses of Proposition* 3.3 *the pair* (Z, Φ_X^a) *is homotopy equivalent to the pair* (X, Φ_X^a). *In particular,* $H_*(X, \Phi_X^a) \cong H_*(Z, \Phi_X^a)$.

Proof. First we observe that Z and $\mathring{Z}$ are positively invariant with respect to the flow φ^t. This follows from our choice of T because an element $u \in \varphi^T(A)$ can leave $\varphi^T(A)$ only via $\varphi^T(B)$ which is already contained in $\mathring{P}\cup(-\mathring{P})\cup\Phi_X^{a-1}$. Next it follows from 3.1 and 3.4 that for every $u \in X$ there exists a time $T(u) \geq 0$ with $\varphi^{T(u)}(u) \in \mathring{Z}$. We choose $\varepsilon(u) > 0$ such that $\varphi^{T(u)}(v) \in \mathring{Z}$ provided that $\|v-u\|_X < \varepsilon(u)$. Then we take a locally finite partition of unity $(\pi_\iota)_{\iota\in I}$ subordinate to the open covering $(K_{\varepsilon(u)}(u) : u \in X)$ of X and choose a family $(u_\iota)_{\iota\in I}$ of points $u_\iota \in X$ with $\operatorname{supp}(\pi_\iota) \subset K_{\varepsilon(u_\iota)}(u_\iota)$. Finally, we define

$$\tau : X \to [0,\infty), \quad \tau(u) := \sum_{\iota\in I} \pi_\iota(u)T(u_\iota),$$

and

$$h : X \times [0,1] \to X, \quad h(u,t) := \varphi^{t\tau(u)}(u).$$

This defines a continuous deformation of (X, Φ_X^a) into (Z, Φ_X^a). □

It is not difficult to see that (Z, Φ_X^a) is a strong deformation retract of (X, Φ_X^a). To see this one checks that the map

$$\overline{\tau} : X \to [0,\infty), \quad \overline{\tau}(u) := \inf\{t \geq 0 : \varphi^t(u) \in Z\}$$

is continuous. Then the map

$$\overline{h}(u,t) := \varphi^{t\overline{\tau}(u)}(u)$$

defines a strong deformation retraction of X into Z. Proposition 3.3 is a consequence of 3.5 and the next lemma.

Lemma 3.6. *Under the hypotheses of Proposition* 3.3 *we have*

$$H_k(Z, \Phi_X^a) \cong \begin{cases} \mathbb{F} & \text{if } k = \mu_0, \\ 0 & \text{if } k \neq \mu_0. \end{cases}$$

Proof. Setting $Z_1 := P \cup (-P) \cup \Phi_X^a$ we first compute $H_*(Z_1, \Phi_X^a)$ and $H_*(Z, Z_1)$ and apply then the long exact sequence of the triple (Z, Z_1, Φ_X^a). If $\mu_\infty = \infty$ then by a radial homotopy Φ_X^a and $Z_1\backslash\{0\}$ are homotopy equivalent, hence $H_*(Z_1\backslash\{0\},\Phi_X^a)\cong\{0\}$. Now we use the long exact sequence of the triple $(Z_1,Z_1\backslash\{0\},\Phi_X^a)$ in order to obtain

$$H_k(Z_1,\Phi_X^a)\cong H_k(Z_1,Z_1\backslash\{0\})\cong\delta_{k1}\mathbb{F}:=\begin{cases}\mathbb{F} & \text{if } k=1,\\ 0 & \text{if } k\neq 1.\end{cases}$$

The last isomorphism follows from the excision property and the homotopy invariance of homology because $H_*(Z_1,Z_1\backslash\{0\})\cong H_*(P\cup(-P),P\cup(-P)\backslash\{0\})\cong H_*(\mathbb{R},\mathbb{R}\backslash\{0\})$.

If $\mu_\infty<\infty$ (but $\mu_\infty\geq 1$ by assumption) we replace the radial homotopy from above by the deformation $(t,u)\mapsto e^{-tL}u$ where $L=\mathrm{Id}-A\in\mathcal{L}(X)$. Let $v_\infty\in P$ with $\|v_\infty\|_E=1$ be the unique normalized eigenvector of L belonging to the smallest eigenvalue of L. Since $\mu_\infty\geq 1$ this smallest eigenvalue is negative. For $u\in P\cup(-P)\backslash\{0\}$ we have $\langle u,v_\infty\rangle\neq 0$, hence $\|e^{-tL}u\|_E\to\infty$ as $t\to\infty$. This implies

$$\frac{d}{dt}\Phi(e^{-tL}u)=-\|Le^{-tL}u\|_E^2+o(\|e^{-tL}u\|^2)\to-\infty\quad\text{as } t\to\infty$$

and therefore $\Phi(e^{-tL}u)\to-\infty$ as $t\to\infty$. This shows that also in the case $1\leq\mu_\infty<\infty$ the sublevel set Φ_X^a is a (strong) deformation retract of $Z_1\backslash\{0\}$. Arguing as above we see that $H_k(Z_1,\Phi_X^a)\cong\delta_{k1}\mathbb{F}$ holds provided $\mu_\infty\geq 1$.

Next we compute $H_*(Z,Z_1)$. If $\mu_0=1$ then the unstable manifold of 0 for the flow φ^t is contained in $\mathring{P}\cup(-\mathring{P})\cup\{0\}$. Therefore Z_1 is a strong deformation retract of Z and $H_*(Z,Z_1)\cong\{0\}$. Now we consider the case $\mu_0>1$. Here we first use the excision property again in order to see that

$$H_*(Z,Z_1)\cong H_*(\varphi^T(A),\varphi^T(A)\cap Z_1).$$

The version of the excision property which we use is the following: $H_*(M_1\cup M_2,M_2)\cong H_*(M_1,M_1\cap M_2)$. This does not hold for arbitrary topological spaces but it does in our case where $M_1,M_2\subset X$ are closed subsets of $M_1\cup M_2$ and deformation retracts of open neighborhoods in X. Now we look at the following part of the long exact sequence of the triple $(\varphi^T(A),\varphi^T(A)\cap Z_1,\varphi^T(A)\cap Z_1\backslash\{0\})$:

$$\begin{aligned}H_k(\varphi^T(A),\varphi^T(A)\cap Z_1\backslash\{0\})&\to H_k(\varphi^T(A),\varphi^T(A)\cap Z_1)\\ &\to H_{k-1}(\varphi^T(A)\cap Z_1,\varphi^T(A)\cap Z_1\backslash\{0\}).\end{aligned}$$

Clearly $\varphi^T(A)\cap Z_1\backslash\{0\}$ is homotopy equivalent to $\varphi^T(B)$, because B is the exit set of A. This yields

$$\begin{aligned}H_k(\varphi^T(A),\varphi^T(A)\cap Z_1\backslash\{0\})&\cong H_k(\varphi^T(A),\varphi^T(B))\\ &\cong H_k(A,B)\cong H_k(D^{\mu_0},S^{\mu_0-1})\cong\delta_{k\mu_0}\mathbb{F}.\end{aligned}$$

Moreover, again by excision we obtain

$$H_{k-1}(\varphi^T(A)\cap Z_1,\varphi^T(A)\cap Z_1\backslash\{0\})\cong H_{k-1}(P\cup(-P),P\cup(-P)\backslash\{0\})\cong\delta_{k2}\mathbb{F}.$$

Putting these facts together we get in the case $\mu_0 > 1$:

$$
\begin{aligned}
H_k(Z, Z_1) &\cong H_k(\varphi^T(A), \varphi^T(A) \cap Z_1) \cong \delta_{k\mu_0}\mathbb{F} \oplus \delta_{k2}\mathbb{F} \\
&\cong \begin{cases} \mathbb{F} \oplus \mathbb{F} & \text{if } \mu_0 = 2 \text{ and } k = 2, \\ \mathbb{F} & \text{if } \mu_0 > 2 \text{ and } k \in \{2, \mu_0\}, \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

The computation also shows that there is a generator of $H_2(Z, Z_1)$ corresponding to the $\delta_{k2}\mathbb{F}$ summand which maps to a generator of $H_1(Z_1, Z_1 \backslash \{0\}) \cong \delta_{k1}\mathbb{F}$ under the boundary homomorphism $H_2(Z, Z_1) \to H_1(Z_1, Z_1 \backslash \{0\})$ of the triple $(Z, Z_1, Z_1 \backslash \{0\})$.

Now we can compute $H_*(Z, \Phi_X^a)$. If $\mu_0 = 1$ we have $H_*(Z, Z_1) \cong \{0\}$ so the long exact sequence of (Z, Z_1, Φ_X^a) yields

$$H_k(Z, \Phi_X^a) \cong H_k(Z_1, \Phi_X^a) \cong \delta_{k1}\mathbb{F} = \delta_{k\mu_0}\mathbb{F}$$

as claimed. It remains to consider the case $\mu_0 > 1$. Again we use the long exact sequence

$$\ldots \to H_k(Z_1, \Phi_X^a) \to H_k(Z, \Phi_X^a) \to H_k(Z, Z_1) \to H_{k-1}(Z_1, \Phi_X^a) \to \ldots$$

Now $H_{k-1}(Z_1, \Phi_X^a) \cong \delta_{k2}\mathbb{F}$, $H_k(Z, Z_1) \cong \delta_{k\mu_0}\mathbb{F} \oplus \delta_{k2}\mathbb{F}$ and the generator of $\delta_{k2}\mathbb{F}$ in $H_2(Z, Z_1)$ maps onto a generator of $H_1(Z_1, \Phi_X^a) \cong H_1(Z_1, Z_1 \backslash \{0\})$. Therefore $H_k(Z, \Phi_X^a) \cong \delta_{k\mu_0}\mathbb{F}$ holds also in the case $\mu_0 > 1$. $\square$

4. Applications

In this section we apply the results of Section 2 to the Dirichlet problem

$$\text{(D)} \qquad -\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and prove generalizations of the theorems mentioned in the introduction. The domain $\Omega \subset \mathbb{R}^N$ will always be bounded with Lipschitz boundary, and the nonlinearity f has to satisfy the condition

(f_1) $f \in \mathcal{C}^1(\mathbb{R})$, $\quad f(0) = 0$.

Different growth conditions for f at infinity will be needed depending on the result. In order to state them let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ be the eigenvalues of the problem

$$\text{(L)} \qquad -\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

For our first result we assume one of the following hypotheses:

(f_2) $\limsup_{|t|\to\infty} f(t)/t < \lambda_1$.

(f_3) $f'(t) \to \omega < \lambda_2$ as $|t| \to \infty$.

Theorem 4.1. *Suppose* (f_1) *holds and* $f'(0) > \lambda_2$.

(a) *If in addition* (f_2) *or* (f_3) *holds then* (D) *has a solution* u_1 *which changes sign. Any positive solution is larger than* u_1, *any negative solution is smaller than* u_1.

(b) *If* (f_2) *holds then* (D) *has a positive solution* u_+ *and a negative* u_-, *hence* $u_-(x) < u_1(x) < u_+(x)$ *for every* $x \in \Omega$. *Any other positive solution is larger than* u_+, *any other negative solution is less than* u_-.

Parts (b) and (c) from Theorem 2 are special cases of 4.1(a), (b), respectively. Next we assume one of the following hypotheses on f for $|t| \to \infty$. As in the introduction F denotes the primitive of f.

(f_4) $f'(t) \to \omega \in \mathbb{R}$ as $|t| \to \infty$. If ω is an eigenvalue of (L) with eigenspace $V = \{u \in C_0^\infty(\Omega) : -\Delta u = \omega u\}$ then

$$\int_\Omega F(u(x))dx \to \infty \quad \text{for } u \in V,\ \|u\| \to \infty,$$

or

$$\int_\Omega F(u(x))dx \to -\infty \quad \text{for } u \in V,\ \|u\| \to \infty.$$

(f_5) There exist constants $R > 0$ and $\theta > 2$ such that

$$0 < \theta F(t) \le t f(t) \quad \text{for } |t| \ge R.$$

(f_6) There exist constants $a > 0$, $p \in (2, 2N/(N-2))$ such that

$$|f'(t)| \le a(|t|^{p-2} + 1) \quad \text{for all } t \in \mathbb{R}.$$

(f_7) There exists $\lambda \in \mathbb{R}$ such that $f'(t) > \lambda$ for all $t \in \mathbb{R}$.

The next result generalizes Theorem 1 from the introduction.

Theorem 4.2. *Suppose* (f_1) *and* (f_4) *hold with* $\omega > \lambda_2$, *or* (f_1), (f_5), (f_6) *and* (f_7) *hold.*

(a) *If* $f'(0) < \lambda_2$ *then* (D) *has a solution* u_1 *which changes sign. This solution has the property that any solution* $u_2 > u_1$ *must be positive and any solution* $u_2 < u_1$ *must be negative.*

(b) *If* $f'(0) < \lambda_1$ *then* (D) *has a positive solution* u_+ *and a negative solution* u_- *with the following property. Any solution* $u_2 < u_+$ *must be negative, and any solution* $u_2 > u_-$ *must be positive.*

It is also possible to prove the existence of a positive and/or a negative solution if $f'(0) = \lambda_1$ provided $F(t) > (\lambda_1/2)t^2$ for $t > 0$ and/or $t < 0$, $|t|$ small. Now we state a generalization of part (a) of Theorem 2 from the introduction.

THEOREM 4.3. *Suppose* (f_1) *holds and* $f'(0)$ *is not an eigenvalue of* (L). *Then* (D) *has a sign changing solution if in addition* (f_5) *and* (f_6) *apply. There also exists a sign changing solution in the case of* (f_4) *provided* ω *is not an eigenvalue of* (L) *and there exists* $k \geq 2$ *such that* $f'(0) < \lambda_k < \omega$ *or* $\omega < \lambda_k < f'(0)$.

The nonresonance conditions on $f'(0)$ and ω in 4.3 can be replaced by conditions on higher order terms of f.

REMARK 4.4. As mentioned in the introduction the existence of the solutions in 4.1 to 4.3 was known, at least in the nonresonance cases; see for instance [AR], [Ch2], [CLL], [H], [Wa]. The existence of a sign changing solution has been proved in [CCN] under much stronger assumptions on f than those of 4.2. In [CCN] it is assumed that $f'(0) < \lambda_1$, that (f_1), (f_6) and a variation of (f_5) holds. Moreover, $f'(t) > f(t)/t$ has to hold for all $t \neq 0$ and $f(t)/t \to \infty$ as $|t| \to \infty$. This implies (f_7). In this situation Castro *et al.* prove the existence of a sign changing solution u_1 with precisely two nodal components, that is, $\Omega \backslash u_1^{-1}(0)$ has exactly two components. We do not know whether a similar result is true in the situation we consider. On the other hand, our results give more information on the relation between the sign changing solution and the positive and negative solutions.

PROOF OF 4.1. Before introducing the variational setting we need to modify the nonlinearity f in the case of (f_2). In [H], Proof of Theorem 8, it is shown that there exists a $\mathcal{C}^1$-modification $\widetilde{f}$ of f satisfying $\widetilde{f}(0) = 0$, $\limsup_{|t|\to\infty} \widetilde{f}(t)/t < \lambda_1$, $\widetilde{f}'(0) > \lambda_2$, and in addition $|\widetilde{f}'(t)| < \lambda$ for all $t \in \mathbb{R}$, and some $\lambda \in \mathbb{R}$. Moreover, solutions of (D) are precisely the solutions of the modified Dirichlet problem

$$-\Delta u = \widetilde{f}(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Thus we may assume that $|f'(t)| < \lambda$ in addition to (f_2). If (f_3) holds then $f'(t)$ is bounded anyway. We fix $\lambda > 0$ with $|f'(t)| < \lambda$ for all $t \in \mathbb{R}$.

Let E be the Sobolev space $H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega uv \, dx.$$

The ordering on E is given by the closed cone $P_E := \{u \in E : u \geq 0 \text{ almost everywhere}\}$. Let X be the Banach space $\mathcal{C}_0^1(\Omega)$ with the usual norm. It is well known that X is dense in E and that $P := X \cap P_E = \{u \in \mathcal{C}_0^1(\Omega) : u \geq 0\}$ has nonempty interior $\mathring{P}$. The element $e \in \mathring{P}$ such that $\langle u, e\rangle > 0$ for all $u \in P \setminus \{0\}$ is the unique normalized positive eigenfunction of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions. Since $f'(t)$ is bounded the energy

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(u) \, dx$$

defines a $\mathcal{C}^2$-functional $\Phi : E \to \mathbb{R}$. The Palais–Smale condition is satisfied in case (f_2) because then Φ is coercive, so Palais–Smale sequences are bounded and have a convergent subsequence since $\nabla\Phi$ is a compact perturbation of the identity. If (f_3) holds with $\omega \neq \lambda_1$ then the Palais–Smale condition can be deduced similarly. If $\omega = \lambda_1$ then the associated eigenspace is one-dimensional and spanned by the positive eigenfunction e. It follows that $\int_\Omega F(te)\,dx \to \infty$ as $|t| \to \infty$, again by (f_3). This is the Landesman–Lazer condition which implies the Palais–Smale condition. It follows from standard regularity theory that critical points of Φ lie in X. Therefore (Φ_1) from Section 2 is satisfied.

Setting $g(t) := f(t) + \lambda t$ and $G(t) := \int_0^t g(s)\,ds$ we can write Φ as

$$\Phi(u) = \frac{1}{2}\|u\|_E^2 - \int_\Omega G(u)\,dx.$$

Since $g'(t) > 0$ by our choice of λ it follows that Φ satisfies (Φ_2) and (Φ_3) from Section 2; see [H]. Moreover, Φ is bounded below and coercive if (f_2) holds. In the case of (f_3) it is easy to see that Φ is asymptotically linear and (Φ_4)(iii) is satisfied. Now Theorem 4.1 follows from Theorem 2.1. The Morse index μ_0 of Φ at 0 is at least 2 because $f'(0) > \lambda_2$. Moreover, in the case of (f_2) we have $\mu_\infty = 0 = \nu_\infty$ because Φ is bounded below. Finally, if (f_3) applies then

$$(\mu_\infty, \nu_\infty) = \begin{cases} (0,0) & \text{if } \omega < \lambda_1, \\ (0,1) & \text{if } \omega = \lambda_1, \\ (1,0) & \text{if } \lambda_1 < \omega < \lambda_2. \end{cases}$$

Therefore $\mu_\infty + \nu_\infty \leq 1$ as required. □

For the proof of 4.2 we can apply Theorem 2.2 as above. We leave it to the reader to check that the hypotheses (Φ_1) to (Φ_4) hold. In fact, (f_4) corresponds to (Φ_4)(iii), and (f_5)–(f_7) to (Φ_4)(ii).

Finally, 4.3 follows from 2.3. □

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