Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 203–223

# MULTIPLE SOLITARY WAVE SOLUTIONS OF NONLINEAR SCHRÖDINGER SYSTEMS

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Abstract. Consider the $N$-coupled nonlinear elliptic system

$$\text{(P)} \qquad \begin{cases} -\Delta U_j + U_j = \mu U_j^3 + \beta U_j \sum\limits_{k\neq j} U_k^2 \quad \text{in } \Omega, \\ U_j > 0 \quad \text{in } \Omega, \quad U_j = 0 \quad \text{on } \partial\Omega,\ j = 1, \ldots, N. \end{cases}$$

where $\Omega$ is a smooth and bounded (or unbounded if $\Omega$ is radially symmetric) domain in $\mathbb{R}^n$, $n \leq 3$. By using a $Z_N$ index theory, we prove the existence of multiple solutions of (P) and show the dependence of multiplicity results on the coupling constant $\beta$.

## 1. Introduction

In this paper, we study the existence and multiplicity of solitary wave solutions of $N$-coupled nonlinear Schrödinger (CNLS) equations (also called Gross–Pitaevskiĭ equations):

$$(1.1) \qquad \begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j|\Phi_j|^2\Phi_j + \Phi_j \displaystyle\sum_{k=1,k\neq j}^{N} \beta_{jk}|\Phi_k|^2, \quad \text{in } \Omega, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad t > 0,\ j = 1, \ldots, N, \end{cases}$$

where $\mu_j > 0$ and $\beta_{jk} = \beta_{kj}$ are constants, and $\Omega \subset \mathbb{R}^n$ is a bounded domain for $n = 1, 2, 3$ or an unbounded radially symmetric domain for $n = 2, 3$. CNLS

2010 *Mathematics Subject Classification.* 35B05, 35B32, 35J50, 58C40.

*Key words and phrases.* Nonlinear Schrödinger system, Nehari manifold, a $Z_N$-index theory.

equations have applications in many physical problems, especially in nonlinear optics. Physically, the solution $\Phi_j$ denotes the $j$-th component of the beam in Kerr-like photorefractive media. The positive constant $\mu_j$ is for self-focusing in the $j$-th component of the beam. The coupling constant $\beta_{jk}$ is the interaction between the $j$-th and the $k$-th component of the beam. When $N = 2$, (1.1) also arises in the Hartree–Fork theory for a double condensate, i.e. a binary mixture of Bose–Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [8]). Physically, $\Phi_1$ and $\Phi_2$ are the corresponding condensates amplitudes, $\mu_j$ and $\beta_{jk}$ are the intraspecies and interspecies scattering lengths. The sign of the scattering length $\beta_{jk}$ determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive.

We consider solitary wave solutions of (1.1) of the form

$$\Phi_j(x,t) = e^{i\lambda_j t}U_j(x), \quad j = 1,\dots,N$$

then system (1.1) is reduced to the following elliptic system about $U_j$

$$\begin{cases} -\Delta U_j + (\lambda_j + V_j)U_j = \mu_j U_j^3 + U_j \displaystyle\sum_{k=1,k\neq j}^{N} \beta_{jk}U_k^2 & \text{in } \Omega, \\ U_j > 0 \quad \text{in } \Omega, \quad U_j = 0 \quad \text{on } \partial\Omega, \ j = 1,\dots,N. \end{cases} \tag{1.2}$$

Recently, system (1.2) has attracted extensive attention of many authors. By imposing conditions on trappings potentials $V_j$ and couplings $\mu_j$, $\beta_{jk}$, some results have been obtained about ground state or bound state solutions (see [1], [2], [4], [5], [7], [11]–[13], [15], [18], [19]), and about semiclassical state or singularly perturbed settings (see [9], [10], [14], [16]), and references therein.

The existence of multiple positive solutions of the fully symmetric case (i.e. $\mu_j = \mu$ for $j = 1,\dots,N$ and $\beta_{ij} = \beta$ for all $1 \leq i, j \leq N$):

$$\begin{cases} -\Delta U_j + U_j = \mu U_j^3 + \beta U_j \displaystyle\sum_{k=1,k\neq j}^{N} U_k^2, & \text{in } \Omega, \\ U_j > 0 \quad \text{in } \Omega, \quad U_j = 0 \quad \text{on } \partial\Omega, \ j = 1,\dots,N \end{cases} \tag{1.3}$$

is also studied by many authors. For work in this respect, we refer to [3], [6], [22], [23], [19].

The current paper is mostly related to and motivated by [6], in which a Lusternik–Schnirelmann type theory and the reflection invariance of corresponding Nehari manifold are used to get multiple solutions of (1.3) for $N = 2$ and $\beta < 0$.

For the general $N$-system, there is a large group action, $S_N$, the permutation group of order $N$ which acts on $\mathbb{R}^N$ by permutating the $N$ components of any vectors. If $(U_1,\dots,U_N)$ is a solution, so is $g(U_1,\dots,U_N)$ for any $g \in S_N$. In general, it is difficult to use this symmetry in relation to critical point theory with symmetry. Our observation in this paper is that a sub-group of $S_N$, $Z_N$

the cyclic group of order $N$, can be employed to effectively deal with multiple critical points of the associated variational formulation.

In doing so, we will define a $Z_N$ index theory, and as a key new ingredient we will construct certain equivariant maps, which can be used to find multiple solutions of (1.3) for $N \geq 3$. We point out that for the variational formulation, we follow closely the work done in [6] for $N = 2$. The novelty of the current paper is our recognition of using a $Z_N$ symmetry instead of $S_N$ symmetry and some technical arguments overcoming difficulties to make use of the $Z_N$ symmetry. Our results are also related to [3], [19] and [23], using different methods. In [19], multiple solutions were constructed by a perturbation type method for the symmetric $N$-system, and in [23] by a heat flow method for the symmetric 2-system. Both [19] and [23] were for radially symmetric cases. In [3], the result of [6], [23] for the symmetric 2-systems on radially symmetric domains were generalized to non-symmetric ($\mu_1 \neq \mu_2$) cases by a local and global bifurcation method.

Let $\sigma$ be the special permutation in $\mathbb{R}^N$ satisfying for $z = (z_1, z_2, \ldots, z_N)$, $\sigma(z) = (z_2, z_3, \ldots, z_N, z_1)$. Then $\sigma$ generates a cyclic group $Z_N$ containing elements id, $\sigma$, $\sigma^2, \ldots, \sigma^{N-1}$. A $Z_N$-orbit of $z$ is the set containing $\sigma^j z$ for $1 \leq j \leq N$.

Our main result is the following theorem:

THEOREM 1.1.

(a) *If $\beta \leq -\mu/(N-1)$, then system* (1.3) *has an infinite sequence of $Z_N$-orbits of solutions.*

(b) *For any positive integer $m$, there exists a $\beta_m \in (-\mu/(N-1), 0)$, such that for $\beta \in (-\mu/(N-1), \beta_m)$, system* (1.3) *has at least $m$ $Z_N$-orbits of solutions.*

REMARK 1.2. System (1.3) is invariant under the action of $S_N$, the permutation group of order $N$. Since each $S_N$ orbit contains at most $(N-1)!$ $Z_N$-orbits, the conclusions can also be stated for $S_N$ orbits of solutions, i.e. for $\beta \leq -\mu/(N-1)$, there are infinitely many $S_N$-orbits of solutions; and for any positive integer $m$, there exists a $\beta'_m \in (-\mu/(N-1), 0)$, such that for $\beta \in (-\mu/(N-1), \beta'_m)$, system (1.3) has at least $m$ $S_N$-orbits or solutions. Actually, we only need to choose $\beta'_m$ close enough to $-\mu/(N-1)$, such that there are at least $m(N-1)!$ $Z_N$-orbits of solutions of (1.3).

This paper is organized as follows. In Section 2, we list and prove some properties of the variational problem of system (1.3), which are closely related to the arguments in [6] for the case $N = 2$. Then the existence of multiple solutions of (1.3) is equivalent to the existence of multiple critical points of the corresponding energy functional. In Section 3, we cite a simpler version of the

$Z_N$-index theory introduced in [21] and examine the $Z_N$ symmetry involved in the problem here. In order to apply the $Z_N$ index theory to construct multiple critical points, we need to construct certain equivariant maps with suitable property. This is done in Section 4 and the proof of Theorem 1.1 will follow.

## 2. Variational structure

Throughout this paper, we denote

$$\overrightarrow{U} = (U_1, \ldots, U_N) \in \mathcal{H} := \begin{cases} [H_0^1(\Omega)]^N & \text{if } \Omega \subset \mathbb{R}^n \text{ is bounded, } n = 1,2,3, \\ [H_{0,r}^1(\Omega)]^N & \text{if } \Omega \subset \mathbb{R}^n \text{ is radially symmetric} \\ & \text{(possibly unbounded), } n = 2,3, \end{cases}$$

where $H_{0,r}^1(\Omega) = \{U \in H_0^1(\Omega) \mid U \text{ is radial function}\}$. For simplicity, we use $\mathcal{H}$ in general, but its definition may vary according to context.

Denote $\overrightarrow{U}_j := (0, \ldots, U_j, \ldots, 0)$ for $U_j \neq 0$, $U_j \in H_0^1(\Omega)$ or $H_{0,r}^1(\Omega)$. Note that $\mathcal{H}$ is a Hilbert space with inner product

$$(\overrightarrow{U}, \overrightarrow{V}) = \sum_{j=1}^N \int_\Omega \nabla U_j \nabla V_j + U_j V_j, \quad \overrightarrow{U}, \overrightarrow{V} \in \mathcal{H}.$$

Denote $\|U\|^2 := \|U\|_{H_0^1}^2 = \int_\Omega (|\nabla U|^2 + U^2)$. Without loss of generality, we assume $\mu = 1$, then the corresponding energy functional of problem (1.3) is

$$E(\overrightarrow{U}) = \frac{1}{2} \sum_{j=1}^N \|U_j\|^2 - \frac{1}{4} \int_\Omega \bigg( \sum_{j=1}^N |U_j^+|^4 \bigg) - \frac{\beta}{4} \int_\Omega \sum_{k \neq j}^N U_j^2 U_k^2 \tag{2.1}$$

where $\beta < 0$. By Sobolev embedding theorem and Proposition B.34 in [17], $E$ is a $C^2$ functional.

We say a critical point $\overrightarrow{U}$ nontrivial, if $U_j \neq 0$, for all $j = 1, \ldots, N$.

Lemma 2.1. *Every nontrivial critical point $\overrightarrow{U}$ of $E$ in $\mathcal{H}$ is a classical solution of* (1.3).

Proof. Let $\overrightarrow{U}$ be a nontrivial critical point of $E$ in $\mathcal{H}$. Then for arbitrary functions $V_j \in H_0^1(\Omega)$, we have

$$(\nabla E(\overrightarrow{U}), \overrightarrow{V}_j) = 0, \quad 1 \leq j \leq N.$$

Thus $\overrightarrow{U}$ is a weak solution of the system

$$\begin{cases} -\Delta U_j + U_j = (U_j^+)^3 + \beta U_j \displaystyle\sum_{k=1, k\neq j}^N U_k^2, & \text{in } \Omega, \\ U_j = 0 \quad \text{on } \partial\Omega, \ j = 1, \ldots, N. \end{cases}$$

Multiplying the $j$-th equation of the above system by $U_j^-$ and integrating over $\Omega$, we get

$$\int_\Omega |\nabla U_j^-|^2 + \int_\Omega \Big(1 - \sum_{k\neq j} \beta U_k^2\Big) |U_j^-|^2 = 0, \quad j = 1, \dots, N.$$

Since $\beta$ is negative, these equations imply $U_j^- = 0$, or equivalently, $U_j \geq 0$ for $j = 1, \dots, N$. By standard elliptic regularity theory, each component $U_j$ of $\overrightarrow{U}$ is $C^2$ function, thus $\overrightarrow{U}$ is a classical solution of (1.3). Then by the strong maximum principle, $U_j > 0$ for $j = 1, \dots, N$. □

Consider the Nehari manifold associated with system (1.3),

$$\begin{aligned}\text{(2.2)}\quad \mathcal{M} = \{\overrightarrow{U} = (U_1, \dots, U_N) \in \mathcal{H} \setminus \{\overrightarrow{0}\} \mid \nabla E(\overrightarrow{U})\overrightarrow{U}_j = 0,\\ \text{and } U_j \neq 0,\ j = 1, \dots, N\}.\end{aligned}$$

Clearly, all nontrivial critical points of $E$ are contained in $\mathcal{M}$. Define functional $F: \mathcal{H} \to \mathbb{R}^N$

$$\text{(2.3)}\qquad F(\overrightarrow{U}) = \begin{pmatrix} F_1(\overrightarrow{U}) \\ \vdots \\ F_N(\overrightarrow{U}) \end{pmatrix} = \begin{pmatrix} \|U_1\|^2 - \beta \int_\Omega U_1^2 \sum_{j\neq 1} U_j^2 - \int_\Omega |U_1^+|^4 \\ \vdots \\ \|U_N\|^2 - \beta \int_\Omega U_N^2 \sum_{j\neq N} U_j^2 - \int_\Omega |U_N^+|^4 \end{pmatrix}.$$

Then $\mathcal{M}$ can be represented as

$$\text{(2.4)}\qquad \mathcal{M} = \{\overrightarrow{U} \in \mathcal{H} : F(\overrightarrow{U}) = \overrightarrow{0},\ U_j \neq 0,\ j = 1, \dots, N\}.$$

Again, by Sobolev embedding theorem and Proposition B.34 in [17], $F$ is a $C^2$ functional. With similar argument used in [6], we see that $\mathcal{M}$ is a $C^2$ manifold in $\mathcal{H}$. Moreover, for any $\overrightarrow{U} \in \mathcal{M}$, we have

$$\begin{aligned}\text{(2.5)}\ E(\overrightarrow{U}) &= \frac{1}{2}\sum_{j=1}^N \|U_j\|^2 - \frac{1}{4}\int_\Omega \Big(\sum_{j=1}^N |U_j^+|^4\Big) - \frac{\beta}{4}\int_\Omega \sum_{k\neq j} U_j^2 U_k^2 \\ &= \frac{1}{2}\sum_{j=1}^N \|U_j\|^2 - \frac{1}{4}\Big(\sum_{j=1}^N \|U_j\|^2 - \beta \int_\Omega \sum_{k\neq j} U_j^2 U_k^2\Big) - \frac{\beta}{4}\int_\Omega \sum_{k\neq j} U_j^2 U_k^2 \\ &= \frac{1}{4}\sum_{j=1}^N \|U_j\|^2\end{aligned}$$

by using (2.1), (2.3) and (2.4).

Lemma 2.2. *Let $E_{\mathcal{M}}$ be the restriction of $E$ to $\mathcal{M}$.*

(a) *If $\overrightarrow{U}$ is a critical point of $E_{\mathcal{M}}$, then $\overrightarrow{U}$ is a nontrivial critical point of $E$.*

(b) *$E_{\mathcal{M}}\colon\mathcal{M}\to\mathbb{R}$ satisfies the Palais–Smale condition.*

Proof. (a) Assume that $\overrightarrow{U}$ is a critical point of $E_{\mathcal{M}}$. Then there exist $N$ Lagrangian multipliers $\lambda_1,\ldots,\lambda_N$ such that

$$\sum_{j=1}^N \lambda_j \nabla F_j(\overrightarrow{U}) = \nabla E(\overrightarrow{U}), \quad \text{in } \mathcal{H}.$$

It is sufficient to show $\lambda_j=0$ for all $j=1,\ldots,N$.

By definition (2.2), we have

$$\sum_{k=1}^N \lambda_k \nabla F_k(\overrightarrow{U})\overrightarrow{U}_j = (\nabla E(\overrightarrow{U}),\overrightarrow{U}_j)=0, \quad j=1,\ldots,N. \tag{2.6}$$

Since $\overrightarrow{U}\in\mathcal{M}$, by (2.3) and (2.4),

$$\begin{aligned}
\partial_{U_j}F_j(\overrightarrow{U})U_j &= 2\|U_j\|^2 - 2\beta\int_\Omega U_j^2\Big(\sum_{k\neq j}U_k^2\Big) - 4\int_\Omega |U_j^+|^4 \\
&= -2\int_\Omega |U_j^+|^4, && j=1,\ldots,N,\\
\partial_{U_j}F_k(\overrightarrow{U})U_j &= -2\beta\int_\Omega U_j^2U_k^2 = \partial_{U_k}F_j(\overrightarrow{U})U_k, && 1\le j\neq k\le N.
\end{aligned}$$

So (2.6) can be written as

$$T_{\overrightarrow{U}}\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_N\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \tag{2.7}$$

where

$$\begin{aligned}
T_{\overrightarrow{U}} &= \begin{pmatrix} \partial_{U_1}F_1(\overrightarrow{U})U_1 & \ldots & \partial_{U_1}F_N(\overrightarrow{U})U_1 \\ \vdots & \ddots & \vdots \\ \partial_{U_N}F_1(\overrightarrow{U})U_N & \ldots & \partial_{U_N}F_N(\overrightarrow{U})U_N \end{pmatrix}\\
&= \begin{pmatrix} -2\displaystyle\int_\Omega |U_1^+|^4 & \ldots & -2\beta\displaystyle\int_\Omega U_1^2U_N^2 \\ \vdots & \ddots & \vdots \\ -2\beta\displaystyle\int_\Omega U_N^2U_1^2 & \ldots & -2\displaystyle\int_\Omega |U_N^+|^4 \end{pmatrix}.
\end{aligned}$$

By (2.3) and Sobolev embedding theorem, there exists a constant $C>0$, such that

$$\|V_j\|^2 \le |V_j|_4^4 \le C\|V_j\|^4, \quad \text{for any } \overrightarrow{V}\in\mathcal{M},\ j=1,\ldots,N.$$

Therefore $\|U_j\| \ge C^{-1/2} > 0$ and

$$\int_\Omega |U_j^+|^4 > -\beta \int_\Omega U_j^2 \sum_{k\neq j} U_k^2, \quad j = 1, \dots, N.$$

These inequalities imply that $T_{\overrightarrow{U}}$ is strictly diagonally dominant. Since all the elements on the major diagonal of $T_{\overrightarrow{U}}$ are negative, $T_{\overrightarrow{U}}$ is negative definite. By (2.7), we have $\lambda_1 = \dots = \lambda_N = 0$, and then conclusion (a) follows.

(b) Let $\{\overrightarrow{U}^k\}_1^\infty = \{(U_1^k, \dots, U_N^k)\}_1^\infty \subset \mathcal{M}$ be a Palais–Smale sequence of $E_{\mathcal{M}}$. Then (2.5) implies that $\{\overrightarrow{U}^k\}_1^\infty$ is bounded in $\mathcal{H}$. Since $\mathcal{H}$ is reflexive, $\{\overrightarrow{U}^k\}_1^\infty$ has a weakly convergent subsequence, still denoted by $\{\overrightarrow{U}^k\}_1^\infty$, which weakly converges to $\overrightarrow{W} \in \mathcal{H}$. By Sobolev embedding, we have

$$|W_j^+|_4^4 = \lim_{k\to\infty} |(U_j^k)^+|_4^4 \ge \liminf_{k\to\infty} \|U_j^k\|^2 - \limsup_{k\to\infty} \beta \int_\Omega \sum_{l\neq j} (U_j^k)^2 (U_l^k)^2 > 0,$$

for $j = 1, \dots, N$.

Thus $W_j^+ \neq 0$ for all $j = 1, \dots, N$. For each $k \ge 1$, there exist Lagrangian multipliers $\lambda_j^k$, such that

$$o(1) = \nabla E_{\mathcal{M}}(\overrightarrow{U}^k) = \nabla E(\overrightarrow{U}^k) - \sum_{j=1}^N \lambda_j^k \nabla F_j(\overrightarrow{U}^k), \quad \text{as } k \to \infty. \tag{2.8}$$

Applying $\nabla E_{\mathcal{M}}(\overrightarrow{U}^k)$ to $\overrightarrow{U}^k$ and using the boundness of $\{\overrightarrow{U}^k\}_1^\infty$ in $\mathcal{H}$, we have

$$\begin{aligned}
o(1) &= \begin{pmatrix} \nabla E(\overrightarrow{U}^k)\overrightarrow{U}_1^k - \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\overrightarrow{U})\right]\overrightarrow{U}_1^k \\ \vdots \\ \nabla E(\overrightarrow{U}^k)\overrightarrow{U}_N^k - \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\overrightarrow{U})\right]\overrightarrow{U}_N^k \end{pmatrix} \\
&= - \begin{pmatrix} \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\overrightarrow{U})\right]\overrightarrow{U}_1^k \\ \vdots \\ \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\overrightarrow{U})\right]\overrightarrow{U}_N^k \end{pmatrix} \\
&= - \begin{pmatrix} \partial_{U_1^k} F_1 U_1^k & \dots & \partial_{U_1^k} F_N U_1^k \\ \vdots & \ddots & \vdots \\ \partial_{U_N^k} F_1 U_N^k & \dots & \partial_{U_N^k} F_N U_N^k \end{pmatrix} \begin{pmatrix} \lambda_1^k \\ \vdots \\ \lambda_N^k \end{pmatrix} \\
&= (-T_{\overrightarrow{W}} + o(1)) \begin{pmatrix} \lambda_1^k \\ \vdots \\ \lambda_N^k \end{pmatrix}, \quad \text{as } k \to \infty.
\end{aligned}$$

By the weakly lower semicontinuity of $\|\cdot\|$,

$$\|W_j\|^2 - \beta \int_\Omega W_j^2 \sum_{l\neq j} W_l^2 \leq \int_\Omega |W_j^+|^4, \quad j = 1, \ldots, N.$$

With similar proof used in (a), $T_{\overrightarrow{W}}$ is negative definite. Therefore

$$\lambda_j^k \to 0, \quad \text{as } k \to \infty, \text{ for all } j = 1, \ldots, N.$$

Note that $\nabla F_j(\overrightarrow{U}^k)$ is bounded in $\mathcal{H}$, then (2.8) implies $\nabla E(\overrightarrow{U}^k) \to \overrightarrow{0}$ as $k \to \infty$. Now for any $\overrightarrow{V} \in \mathcal{H}$,

$$(\nabla E(\overrightarrow{U}^k), \overrightarrow{V}) \to 0, \quad \text{as } k \to \infty.$$

But we also have $\overrightarrow{U}^k \rightharpoonup \overrightarrow{W}$, so $(\nabla E(\overrightarrow{W}), \overrightarrow{V}) = 0$ for any $\overrightarrow{V} \in \mathcal{H}$, i.e. $\overrightarrow{W}$ is a weak solution of

$$\begin{cases} -\Delta W_j + W_j = (W_j^+)^3 + \beta W_j \displaystyle\sum_{l\neq j} W_l^2, & \text{in } \Omega, \\ W_j = 0 & \text{on } \partial\Omega, \ j = 1, \ldots, N. \end{cases}$$

Multiplying the first equation by $W_1$ and integrating over $\Omega$, we get

$$\begin{aligned} \|W_1\|^2 &= |W_1^+|_4^4 + \beta \int_\Omega W_1^2 \sum_{j\neq 1} W_j^2 \\ &= \lim_{k\to\infty} \bigg( |(U_1^k)^+|_4^4 + \beta \int_\Omega (U_1^k)^2 \sum_{j\neq 1} (U_j^k)^2 \bigg) = \lim_{k\to\infty} \|U_1^k\|^2 \end{aligned}$$

thus $U_1^k \to W_1$ strongly in $H_0^1(\Omega)$. Similarly, $U_j^k \to W_j$ strongly in $H_0^1(\Omega)$ for $j = 2, \ldots, N$. So $\overrightarrow{U}^k \to \overrightarrow{W}$ strongly in $\mathcal{H}$. Therefore $E_{\mathcal{M}}$ satisfies the Palais–Smale condition. □

Now, we consider the level sets of $E$ on Nehari manifold

$$\mathcal{M}^c = \{\overrightarrow{U} \in \mathcal{M} : E(\overrightarrow{U}) \leq c\}, \quad c \in \mathbb{R}$$

and the sets of critical points of $E$ on $\mathcal{M}^c$

$$\begin{aligned} K_c &= \{\overrightarrow{U} \in \mathcal{M} \mid E(\overrightarrow{U}) = c, \ \nabla E(\overrightarrow{U}) = 0\} \\ &= \{\overrightarrow{U} \in \mathcal{M} \mid E_{\mathcal{M}}(\overrightarrow{U}) = c, \ \nabla E_{\mathcal{M}}(\overrightarrow{U}) = 0\}. \end{aligned}$$

It is easy to see that $\mathcal{M}, \mathcal{M}_c$ and $K_c$ are all invariant under the action of $S_N$. Especially they are invariant under the action of the generator of $Z_N$:

$$\sigma : \mathcal{H} \to \mathcal{H}, \quad (U_1, \ldots, U_N) \mapsto \sigma(U_1, \ldots, U_N) = (U_2, \ldots, U_N, U_1).$$

Let $N = p_1^{t_1} p_2^{t_2} \ldots p_s^{t_s}$ be the prime factorization of $N$, where $1 < p_1 < \ldots < p_s$ are prime numbers and $t_1, \ldots, t_s$ are positive integers. Let

$$1 = q_0 < q_1 < \ldots < q_a < N, \quad \text{for some integer } a \geq 0$$

be all the distinct factors of $N$. Correspondingly, define $N = N_0 > N_1 > \ldots > N_a > 1$ by $N_b = N/q_b$ for $0 \le b \le a$.

Define the least energy of $E$ on the sets of fixed points of $\sigma^{q_b}$,

$$c^{q_b}(\beta) := \inf \Big\{ E(\overrightarrow{U}) \mid \overrightarrow{U} \in \mathcal{M},\ \sigma^{q_b}(\overrightarrow{U}) = \overrightarrow{U} \Big\} \quad b = 0, \ldots, a$$

and $c^{q_b}(\beta) = \infty$ if $\sigma^{q_b}$ has no fixed point on $\mathcal{M}$. The following lemma shows the dependance of $c^{q_b}(\beta)$ on $\beta$.

Lemma 2.3. *$c^{q_b}(\beta) = \infty$ for $\beta \le -1/(N_b - 1)$, and $\lim_{\beta \searrow -1/(N_b-1)} c^{q_b}(\beta) = \infty$ for $0 \le b \le a$.*

Proof. If $\beta \le -1/(N_b - 1)$ and $\sigma^{q_b}(\overrightarrow{U}) = \overrightarrow{U}$, i.e. $\overrightarrow{U} = (U_1, \ldots, U_{q_b}, \ldots, U_1, \ldots, U_{q_b})$, then we have

$$\begin{aligned} \|U_j\|^2 &= (1 + \beta(N_b - 1))|U_j^+|_4^4 + \beta(N_b - 1)|U_j^-|^4 + \beta N_b \int \sum_{k \ne j}^{q_b} U_k^2 U_j^2 \\ &\le (1 + \beta(N_b - 1))|U_j^+|_4^4 + \beta(N_b - 1)|U_j^-|^4 \le 0 \end{aligned}$$

which implies $\|U_j\| = 0$ for all $j = 1, \ldots, q_b$, therefore $\|U\| = 0$. But $0 \notin \mathcal{M}$, so $\sigma^{q_b}$ has no fixed point on $\mathcal{M}$. By definition, $c^{q_b}(\beta) = \infty$.

If $-1/(N_b - 1) < \beta < 0$ and $\sigma^{q_b}(\overrightarrow{U}) = \overrightarrow{U}$, then

$$\begin{aligned} \|U_1\|^2 &= (1 + \beta(N_b - 1))|U_1^+|_4^4 + \beta(N_b - 1)|U_1^-|^4 + \beta N_b \int \sum_{k \ne 1}^{q_b} U_k^2 U_1^2 \\ &\le (1 + \beta(N_b - 1))|U_1^+|_4^4 + \beta(N_b - 1)|U_1^-|^4 \le C(1 + \beta(N_b - 1))\|U_1\|^4 \end{aligned}$$

where $C > 0$ is the embedding constant corresponding to $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Since $C$ does not depend on $\beta$, we have $\|U_1\|^2 \ge 1/(C(1 + (N_b - 1)\beta))$. Then (2.5) gives us $E(\overrightarrow{U}) \ge \|U_1\|^2 \ge 1/(C(1 + (N_b - 1)\beta))$. Thus $E(\overrightarrow{U}) \to \infty$ as $\beta \to -1/(N_b - 1)$. □

Also, we have the $\sigma$-equivariant deformation lemma.

Lemma 2.4. *Let $c \in \mathbb{R}$, and let $\mathcal{N} \subset \mathcal{M}$ be a relatively open and $\sigma$-invariant neighbourhood of $K_c$. Then there exists $\varepsilon > 0$ and a $C^1$-deformation $\eta\colon [0,1] \times \mathcal{M}^{c+\varepsilon} \setminus \mathcal{N} \to \mathcal{M}^{c+\varepsilon}$ such that for all $\overrightarrow{U} \in \mathcal{M}^{c+\varepsilon} \setminus \mathcal{N}$ and $t \in [0,1]$,*

$$\eta(0, \overrightarrow{U}) = \overrightarrow{U}, \quad \eta(1, \overrightarrow{U}) \in \mathcal{M}^{c-\varepsilon} \quad \text{and} \quad \sigma[\eta(t, \overrightarrow{U})] = \eta(t, \sigma\overrightarrow{U}) \tag{2.9}$$

Proof. Since $E_{\mathcal{M}}$ satisfies Palais–Smale condition, $K_c$ is relatively compact in $\mathcal{M}$. Note that $E$ and $F$ are $C^2$ functionals, then (2.8) implies that $\nabla E_{\mathcal{M}}$ is $C^1$ smooth. There exists $\varepsilon > 0$ and $\delta > 0$, such that

$$|\nabla E_{\mathcal{M}}(\overrightarrow{U})| \ge \sqrt{\delta}, \quad \text{for any } \overrightarrow{U} \in \mathcal{M}^{c+\varepsilon} \setminus (\mathcal{M}^{c-\varepsilon} \cup \mathcal{N}).$$

Consider the descending flow $\eta\colon [0,1]\times\mathcal{M}^{c+\varepsilon}\setminus\mathcal{N}\to\mathcal{M}^{c+\varepsilon}$ determined by the following initial value problem

$$\begin{cases} \dfrac{d\eta(t,\overrightarrow{U})}{dt} = -\dfrac{2\varepsilon}{\delta}\nabla E_{\mathcal{M}}(\eta(t,\overrightarrow{U})), \\ \eta(0,\overrightarrow{U}) = \overrightarrow{U}. \end{cases}$$

Claim. $\eta$ is a deformation satisfying all requirements of the lemma.

First, $\eta$ is a $C^1$ deformation since $\nabla E_{\mathcal{M}}$ is $C^1$ vector field on $\mathcal{M}$. Next, if $\overrightarrow{U}\in\mathcal{M}^{c-\varepsilon}$, then by the descending feature of the deformation flow, $\eta(1,\overrightarrow{U})\in\mathcal{M}^{c-\varepsilon}$. If $\overrightarrow{U}\in\mathcal{M}^{c+\varepsilon}\setminus(\mathcal{M}^{c-\varepsilon}\cup\mathcal{N})$, then

$$\begin{aligned} E(\eta(1,\overrightarrow{U})) &= \int_0^1 -\frac{2\varepsilon}{\delta}|\nabla E(\eta(t,\overrightarrow{U}))|^2\,dt + E(\eta(0,\overrightarrow{U})) \\ &\leq -2\varepsilon + E(\eta(0,\overrightarrow{U})) \leq c-\varepsilon. \end{aligned}$$

Using the fact that $E_{\mathcal{M}}$ and $\nabla E_{\mathcal{M}}$ are $\sigma$ invariant, we see that $\sigma\eta(t,\overrightarrow{U})$ and $\eta(t,\sigma\overrightarrow{U})$ satisfy the same Cauchy problem. Then by the uniqueness of solution, we have $\sigma[\eta(t,\overrightarrow{U})]=\eta(t,\sigma\overrightarrow{U})$. Thus the claim holds and the lemma follows. □

## 3. A $Z_N$-index

We define an index associated with the cyclic group $Z_N$.

Definition 3.1. For any closed $\sigma$-invariant subset $A\subset\mathcal{M}$, define index $\gamma(A)$ as the smallest $m\in\mathbb{N}\cup\{0\}$ such that there exists a continuous map $h\colon A\to\mathbb{C}^m\setminus\{0\}$ satisfying

$$h(\sigma u) = e^{i2\pi/N}h(u). \tag{3.1}$$

If there is no such a map, set $\gamma(A)=\infty$. Define $\gamma(\emptyset)=0$.

In particular, if $A$ contains a fixed point of $\sigma^{q_b}$ for $0\leq b\leq a$, then $\gamma(A)=\infty$. The following lemma lists some properties of this index, which corresponds to a special case of the results in [21].

Lemma 3.2. *Let $A,B\subset\mathcal{M}$ be closed and $\sigma$-invariant.*

(a) *If $A\subset B$, then $\gamma(A)\leq\gamma(B)$;*
(b) $\gamma(A\cup B)\leq\gamma(A)+\gamma(B)$.
(c) *If $g\colon A\to\mathcal{M}$ is continuous and $\sigma$-equivariant, i.e.*

$$g(\sigma(\overrightarrow{U})) = \sigma g(\overrightarrow{U}), \quad \textit{for all } \overrightarrow{U}\in A$$

*then $\gamma(A)\leq\gamma(\overline{g(A)})$.*

*If $A$ does not contain fixed point of $\sigma^{q_b}$ for $0 \le b \le a$:*

(d) *$\gamma(A) > 1$ implies that $A$ is an infinite set;*

(e) *if $A$ is compact, then $\gamma(A) < \infty$, and there exists a relatively open and $\sigma$-invariant neighbourhood $\mathcal{N}$ of $A$ in $\mathcal{M}$ such that $\gamma(A) = \gamma(\overline{\mathcal{N}})$.*

*Finally,*

(f) *if $S$ is the boundary of a bounded and $\sigma$-invariant neighbourhood of zero in a $m$-dimensional complex normed vector space and $\Psi\colon S \to \mathcal{M}$ is continuous map satisfying $\Psi(e^{i2\pi/N}\overrightarrow{U}) = \sigma(\Psi(\overrightarrow{U}))$, then $\gamma(\Psi(S)) \ge m$.*

Proof. (a) Without loss of generality, assume $\gamma(B) = m < \infty$. By definition, there exists a continuous map $h\colon B \to \mathbb{C}^m \setminus \{0\}$ with

$$h(\sigma(\overrightarrow{U})) = e^{i2\pi/N} h(\overrightarrow{U}) \quad \text{for all } \overrightarrow{U} \in B$$

The restriction of $h$ on $A$ is also a continuous map satisfying (3.1). Then (a) follows from Definition 3.1.

(b) Suppose $\gamma(A) = m_1$ and $\gamma(B) = m_2$. Then there exist continuous maps $\phi \in C(A, \mathbb{C}^{m_1} \setminus \{0\})$ and $\psi \in C(B, \mathbb{C}^{m_2} \setminus \{0\})$, both satisfying (3.1). By the Tietze Extension Theorem, there are continuous maps $\widehat{\phi} \in C(E, \mathbb{C}^{m_1})$ and $\widehat{\psi} \in C(E, \mathbb{C}^{m_2})$ such that $\widehat{\phi}|_A = \phi$ and $\widehat{\psi}|_B = \psi$. Replacing $\widehat{\phi}, \widehat{\psi}$ by

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{-i2j\pi/N}\widehat{\phi}(\sigma^j \overrightarrow{U}), \quad \frac{1}{N}\sum_{j=0}^{N-1} e^{-i2j\pi/N}\widehat{\psi}(\sigma^j \overrightarrow{U})$$

if it is necessary, assume that $\widehat{\phi}, \widehat{\psi}$ satisfy (3.1). Set $h = (\widehat{\phi}, \widehat{\psi})$. Then $h \in C(A \cup B, \mathbb{C}^{m_1+m_2} \setminus \{0\})$ satisfies (3.1). According to definition, $\gamma(A \cup B) \le m_1 + m_2 = \gamma(A) + \gamma(B)$.

(c) Without loss of generality, assume $\gamma(\overline{g(A)}) = m < \infty$. By Definition 3.1, there exists a continuous map $\widetilde{g}\colon (\overline{g(A)}) \to \mathbb{C}^m \setminus \{0\}$, satisfying (3.1). Then the composite map

$$\widetilde{g} \circ g\colon A \to \mathbb{C}^m \setminus \{0\}$$

also satisfies (3.1). Therefore,

$$\gamma(A) \le m = \gamma(\overline{g(A)}).$$

(d) If $A \subset \mathcal{M}$ is a finite set, then there exists $m \in \mathbb{N}$ such that

$$A = \left\{\overrightarrow{U}^1, \dots, \overrightarrow{U}^m, \sigma\overrightarrow{U}^1, \dots, \sigma\overrightarrow{U}^m, \dots\dots, \sigma^{N-1}\overrightarrow{U}^1, \dots, \sigma^{N-1}\overrightarrow{U}^m\right\}$$

where $\overrightarrow{U}^k \in \mathcal{M}$ $(k = 1, \dots, m)$ are $m$ $N$-vectors. Define map $h\colon A \to \mathbb{C}^1 \setminus \{0\}$ as

$$h(\sigma^j \overrightarrow{U}^k) = e^{i2(j+1)\pi/N}, \quad j = 0, \dots, N-1,\ k = 1, \dots, m.$$

It is easy to see that $h$ is continuous and satisfies (3.1). By definition, we have $\gamma(A) = 1$.

(e) If $A$ is compact and $\overrightarrow{0} \notin A$, then there exists $\rho > 0$ such that $A \cap \mathcal{B}_\rho(\overrightarrow{0}) = \emptyset$. The cover

$$\Big\{\widetilde{\mathcal{B}}_\rho(\overrightarrow{U}) = \bigcup_{j=0}^{N-1} \mathcal{B}_\rho(\sigma^j \overrightarrow{U})\Big\}_{\overrightarrow{U} \in A}$$

admits a finite sub-cover $\{\widetilde{\mathcal{B}}_\rho(\overrightarrow{U}^1), \ldots, \widetilde{\mathcal{B}}_\rho(\overrightarrow{U}^m)\}$. By choosing $\rho > 0$ small enough, we may assume $\widetilde{\mathcal{B}}_\rho(\sigma^k \overrightarrow{U}) \cap \widetilde{\mathcal{B}}_\rho(\sigma^l \overrightarrow{U}) = \emptyset$ if $1 \leq k \neq l \leq m$. Let $\{\phi_k\}_1^m$ be a partition of unity on $A$ subordinate to $\{\widetilde{\mathcal{B}}_\rho(\overrightarrow{U}^k)\}_1^m$, i.e. $\phi_k \in C(A)$ with $\operatorname{supp}(\phi_k) \subset \widetilde{\mathcal{B}}_\rho(\overrightarrow{U}^k)$, and $0 \leq \phi_k \leq 1$, $\sum_{k=1}^m \phi_k(\overrightarrow{U}) = 1$, for all $\overrightarrow{U} \in A$. Replacing $\phi_k$ by

$$\frac{1}{N} \sum_{j=0}^{N-1} \phi_k(\sigma^j \overrightarrow{U}), \quad 1 \leq k \leq m$$

if it is necessary, we may assume that $\phi_k$ is $\sigma$-invariant. Then for each $k$, define $h_k \colon A \to \mathbb{C}$ as

$$h_k(\overrightarrow{U}) = \begin{cases} e^{i2j\pi/N} \phi_k(\overrightarrow{U}) & \text{if } \overrightarrow{U} \in \mathcal{B}_\rho(\sigma^j \overrightarrow{U}^k),\ j = 0, \ldots, N-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $h := (h_1, \ldots, h_m) \colon A \to \mathbb{C}^m \setminus \{\overrightarrow{0}\}$ is continuous and satisfies (3.1). By Definition 3.1, $\gamma(A) \leq m < \infty$.

Assume that $A$ is compact, $\overrightarrow{0} \notin A, \gamma(A) = m < \infty$ and let $h \in C(A, \mathbb{C}^m \setminus \{0\})$ be a continuous map with property (3.1). We may extend $h$ such that $h \in C(\mathcal{M}, \mathbb{C}^m)$. Since $A$ is compact, its image under continuous map $h$ is also compact. Then there exists an open neighbourhood $\widetilde{\mathcal{N}}$ of $h(A)$ compactly contained in $\mathbb{C}^m \setminus \{0\}$, satisfying

$$e^{i2\pi/N} I_{m\times m} \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}$$

where $I_{m\times m}$ is the $m$ by $m$ identity matrix. Define $\overline{\mathcal{N}} = h^{-1}(\widetilde{\mathcal{N}})$. By construction, $\overrightarrow{0} \notin h(\overline{\mathcal{N}})$ and $\gamma(\overline{\mathcal{N}}) \leq m$. On the other hand, $\gamma(A) \leq \gamma(\overline{\mathcal{N}})$ holds by using (a). Hence $\gamma(A) = \gamma(\overline{\mathcal{N}})$

(f) If $\gamma(\Psi(S)) \leq m-1$, there exists a continuous map $h \colon \Psi(S) \to \mathbb{C}^{m-1} \setminus \{0\}$ with property (3.1). Then $h \circ \Psi \colon S \to \mathbb{C}^{m-1} \setminus \{0\}$ is continuous and has property

$$(h \circ \Psi)(e^{i2\pi/N} z) = e^{i2\pi/N} (h \circ \Psi)(z), \quad \text{for all } z \in S.$$

Applying the $Z_p$-Borsuk–Ulam theorem given in [20, Theorem 2], it is easy to see that (using the notations from [20])

$$m[t_1 - (t_1 \wedge l_1)] < r_1 - (t_1 \wedge l_1 \wedge r_1)$$

where $t_1 \wedge l_1 = \min\{t_1, l_1\}$, $t_1 = 0$ and $l_1 = r_1 = 1$. Since $m-1 < m$, the theorem guarantees $0 \in h(\Psi(S))$, which is a contradiction. Therefore $\gamma(\Psi(S)) \geq m$. $\square$

## 4. Proof of Theorem 1.1

Let $\mathbb{S}^{2m-1}$ be the unit sphere in $\mathbb{C}^m$. To use the $Z_N$-index given in Section 3, we need to construct a continuous map from $\mathbb{S}^{2m-1}$ to $\mathcal{M}$ with

$$\psi(e^{i2\pi/N}U) = \sigma\psi(U), \quad \text{for all } U \in \mathbb{S}^{2m-1}.$$

Proposition 4.1 presents the construction for bounded $\Omega$ and Proposition 4.2 presents the construction corresponding to radially symmetric $\Omega$.

Proposition 4.1. *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $n \le 3$. Then for any $m \ge 1$, there exists a continuous map $\psi\colon \mathbb{S}^{2m-1} \to \mathcal{M}$, such that*

$$\psi(e^{i\frac{2\pi}{N}}U) = \sigma\psi(U), \quad \textit{for all } U \in \mathbb{S}^{2m-1}.$$

Proof. Without loss of generality, assume that the origin is an interior point of $\Omega$.

If $n = 2$, we use polar coordinate system. Let $m \ge 1$ be fixed. There exists $\rho_0 > 0$, such that

$$D := \{(\rho,t) \mid 0 \le \rho < \rho_0,\ 0 \le t < 2\pi\} \subset \Omega.$$

Divide $D$ into $N$ parts $D_1, \ldots, D_N$, where

$$D_j = \left\{(\rho,t) \in D \;\middle|\; \frac{2\pi(j-1)}{N} \le t < \frac{2\pi j}{N},\ 0 \le \rho < \rho_0\right\}, \quad j = 1, \ldots, N.$$

Choose $0 < \rho_1 < \ldots < \rho_m < \rho_{m+1} := \rho_0$, and $m$ functions $U_1^1, \ldots, U_1^m \in H_0^1(\Omega)$, such that the support of $U_1^k$ is contained in $D_1 \cap \{(\rho,t) \mid \rho_k < \rho < \rho_{k+1}\}$. Define

$$\begin{aligned}(4.1)\quad U_j^k(\rho,t) &= U_1^k(\rho, t - 2(j-1)\pi/N),\\ &\qquad 0 \le \rho < \rho_0,\ \frac{2(j-1)\pi}{N} \le t < \frac{2j\pi}{N}, \quad j = 2, \ldots, N.\end{aligned}$$

and let $U^k(\rho,t) = \sum_{j=1}^N U_j^k(\rho,t)$, $k = 1, \ldots, m$.

According to the definition, $U_i^k U_j^k = 0$ if $i \ne j$, $k = 1, \ldots, m$, and $U_i^k U_j^l = 0$ for $k \ne l$, $i,j = 1, \ldots, N$. We define $\mathcal{C}^m$ to be the space spanned by $U^k$, i.e.

$$\mathcal{C}^m = \left\{ \sum_{k=1}^m r_k e^{i\theta_k} U^k \;\middle|\; r_k \in \mathbb{R}^+,\ \theta_k \in [0,2\pi) \text{ and } \theta_k = 0 \text{ if } r_k = 0;\ k = 1, \ldots, m \right\}.$$

Thus $\mathcal{C}^m$ is identified as a $2m$ dimensional linear subspace of $\mathcal{H}$. And the unit sphere in $\mathcal{C}^m$ can be represented as

$$\mathbb{S}^{2m-1} = \left\{ \sum_{k=1}^m r_k e^{i\theta_k} U^k \in \mathcal{C}^m \;\middle|\; \sum_{k=1}^m r_k^2 = 1 \right\}. \tag{4.2}$$

Define map $\psi\colon \mathbb{S}^{2m-1} \to \mathcal{M}$

$$\psi\bigg(\sum_{k=1}^{m} r_k e^{i\theta_k} U^k(\rho,t)\bigg) = (U_1^*(\rho,t),\dots,U_N^*(\rho,t)) \tag{4.3}$$

where

$$U_j^*(\rho,t) = \left(\frac{\bigg\|\sum_{k=1}^{m} r_k U_j^k(\rho,t-\theta_k)\bigg\|}{\bigg|\sum_{k=1}^{m} r_k U_j^k(\rho,t-\theta_k)\bigg|_4^2}\right)\bigg|\sum_{k=1}^{m} r_k U_j^k(\rho,t-\theta_k)\bigg|, \quad j=1,\dots,N.$$

Then it is easy to see that $\psi$ is continuous, $U_j^* \neq 0$ for $j=1,\dots,N$, $U_i^* U_j^* = 0$ for $i \neq j$, and

$$\begin{aligned}\psi\bigg(e^{i2\pi/N}\sum_{k=1}^{m} r_k e^{i\theta_k} U^k(\rho,t)\bigg) &= \psi\bigg(\sum_{k=1}^{m} r_k e^{i(\theta_k+2\pi/N)} U^k(\rho,t)\bigg)\\ &= (U_2^*(\rho,t),\dots,U_N^*(\rho,t),U_1^*(\rho,t)\bigg) = \sigma\psi\bigg(\sum_{k=1}^{m} r_k e^{i\theta_k} U^k(\rho,t)\bigg).\end{aligned}$$

If $n=3$, we choose a cylindrical coordinates and proper constants $\rho_0 > 0$, $h_0 > 0$ such that

$$E := \{(\rho,t,h) \mid 0 \le \rho < \rho_0,\ 0 \le t < 2\pi,\ |h| \le h_0\} \subset \Omega.$$

Then we divide $E$ into $N$ parts along $t$ direction in the same manner as we did above, and consider the functions of the form

$$\widetilde{U}_j^k(\rho,t,h) = \phi^k(h) U_j^k(\rho,t)$$

where $\phi^k(h)$ is continuous function with $\operatorname{supp}\phi \subset (-h_0,h_0)$, and $U_j^k$ is defined in (4.1).

For $n=1$, we divide a path connected subset of $\Omega$, say $E$, into $N$ parts in direction $t$ (a scaling of $E$ may be required), as we did above. Then (4.1)–(4.3) will give the corresponding construction for the case $n=1$ and 3. $\square$

Proposition 4.2. *Let $\Omega$ be a radially symmetric domain in $\mathbb{R}^n$, $n=2,3$. Then for any $m \ge 1$, there exists a continuous map $\psi\colon \mathbb{S}^{2m-1} \to \mathcal{M}$, such that*

$$\psi(e^{i2\pi/N}U) = \sigma\psi(U), \quad \textit{for all } U \in \mathbb{S}^{2m-1}.$$

Proof. Divide $\Omega$ into $N$ radially symmetric open subsets $\Omega_j$, $j=1,\dots,N$ such that

$$\Omega = \bigcup_{j=1}^{N} \Omega_j, \quad \Omega_j \cap \Omega_k = \emptyset, \quad \text{if } j \neq k.$$

Denote by $\mathcal{O} = \mathbb{S}^1 \times \Omega$ with $\mathbb{S}^1$ being the unit circle. Choose functions $U_j: \mathcal{O} \to \mathbb{R}$, $j = 1, \ldots, N$, satisfying following conditions:

(a) $U_j(t,x) = U_j(t,|x|)$ for all $(t,x) \in \mathbb{S}^1 \times \Omega$, and $U_j \in C^1(\mathcal{O})$;
(b) $\operatorname{supp} U_j(\,\cdot\,,\,\cdot\,) \subset \mathbb{S}^1 \times \Omega_j$;
(c) $\operatorname{supp} U_j(t+2i\pi/N,\,\cdot\,) \cap \operatorname{supp} U_j(t+2l\pi/N,\,\cdot\,) = \emptyset$, for all $t$ and $1 \le i \ne l \le N$;
(d) for any $t$, $\sum_{j=1}^N |U_j(t,\,\cdot\,)|_{L^4(\Omega)} \ne 0$.

Let $r = |x|$, then $U_j(t,r) = U_j(t,|x|)$.

Let $m \ge 1$ be fixed. We choose $m$ functions $U^k(t,r) = \sum_{j=1}^N U_j^k(t,r)$, $k = 1, \ldots, m$, where $U_1^k, \ldots, U_N^k$ satisfy the above conditions (a)–(d), and $\operatorname{supp} U^k(\,\cdot\,,\,\cdot\,) \cap \operatorname{supp} U^l(\,\cdot\,,\,\cdot\,) = \emptyset$ if $k \ne l$ (which can be accomplished by slicing $\mathcal{O}$ further in $r$ direction into annuli). Then the following space is identified as an $m$-dimensional complex space

$$\mathcal{C}^m = \bigg\{ \sum_{k=1}^m d_k e^{i\theta_k} U^k \;\bigg|\; d_k \in \mathbb{R}^+, \theta_k \in [0,2\pi) \text{ and } \theta_k = 0 \text{ if } d_k = 0,\ k = 1, \ldots, m \bigg\}$$

and the unit sphere in $\mathcal{C}^m$ is

$$\mathbb{S}^{2m-1} = \bigg\{ \sum_{k=1}^m d_k e^{i\theta_k} U^k \in \mathcal{C}^m \;\bigg|\; \sum_{k=1}^m d_k^2 = 1 \bigg\}. \tag{4.4}$$

For any vector $Y = \sum_{k=1}^m d_k e^{i\theta_k} U^k$ in $\mathbb{S}^{2m-1}$ we set $V(Y): \mathcal{O} \to \mathbb{R}$ as

$$V(Y)(t,r) = \frac{\Big\| \sum_{k=1}^m d_k U^k(t+\theta_k, r) \Big\|}{\Big| \sum_{k=1}^m d_k U^k(t+\theta_k, r) \Big|_4^2} \Big| \sum_{k=1}^m d_k U^k(t+\theta_k, r) \Big|.$$

Now define map $\psi: \mathbb{S}^{2m-1} \to \mathcal{M}$

$$\psi(Y) = \psi\bigg( \sum_{k=1}^m d_k e^{i\theta_k} U^k \bigg) = (V(Y)(t_1^*,\,\cdot\,), \ldots, V(Y)(t_N^*,\,\cdot\,)) \tag{4.5}$$

where for $j = 1, \ldots, N$, $t_j^* = 2(j-1)\pi/N$. Then $\psi$ is continuous,

$$V(Y)(t_i^*,\,\cdot\,) V(Y)(t_j^*,\,\cdot\,) = 0 \quad \text{for } i \ne j$$

and satisfies

$$\begin{aligned}\psi(e^{i2\pi/N}Y) =&\psi\bigg(e^{i2\pi/N}\sum_{k=1}^m d_k e^{i\theta_k}U^k\bigg) = \psi\bigg(\sum_{k=1}^m d_k e^{i(\theta_k+2\pi/N)}U^k\bigg)\\ =&(V(e^{i2\pi/N}Y)(t_1^*,\,\cdot\,),\ldots,V(e^{i2\pi/N}Y)(t_N^*,\,\cdot\,))\\ =&\bigg(V(Y)\bigg(t_1^*+\frac{2\pi}{N},\,\cdot\,\bigg),\ldots,V(Y)\bigg(t_N^*+\frac{2\pi}{N},\,\cdot\,\bigg)\bigg)\\ =&(V(Y)(t_2^*,\,\cdot\,),\ldots,V(Y)(t_N^*,\,\cdot\,),V(Y)(t_1^*,\,\cdot\,))\\ =&\sigma\psi\bigg(\sum_{k=1}^m d_k e^{i\theta_k}U^k\bigg) = \sigma\psi(Y).\end{aligned}$$

□

Remark 4.3. The method used in Proposition 4.2 also works for bounded domains, for which we only need to consider a ball entirely contained in $\Omega$.

Now, we construct multiple critical points for $E_{\mathcal M}$ by using the $Z_N$-index theory. Define the Lusternik–Schnirelman type levels on $\mathcal M$ as

$$c_k := \inf\{c\in\mathbb R \mid \gamma(\mathcal M^c)\ge k\},\quad k=1,2,\ldots.$$

We give an estimate of the index $\gamma$ near the critical levels.

Lemma 4.4. *For any $c<\min_{0\le b\le a}\{c^{q_b}(\beta)\}$, the $Z_N$-index of $K_c$ is finite, i.e. $\gamma(K_c)<\infty$. And there exists $\varepsilon>0$ such that*

$$\gamma(\mathcal M^{c+\varepsilon})\le\gamma(\mathcal M^{c-\varepsilon})+\gamma(K_c).$$

Proof. Since $E_{\mathcal M}$ satisfies the Palais–Smale condition, the set $K_c$ is compact. By the definition of $c^{q_b}(\beta)$ and the assumption $c<\min_{0\le b\le a}\{c^{q_b}(\beta)\}$, there is no fixed point of $\sigma^{q_b}$ in $K_c$ for $0\le b\le a$. By Lemma 3.2(e), we have $\gamma(K_c)<\infty$ and a relatively open $\sigma$-invariant neighbourhood $\mathcal N$ of $K_c$ such that $\gamma(\overline{\mathcal N})=\gamma(K_c)$.

For $\varepsilon>0$ small, let $\eta\colon[0,1]\times\mathcal M^{c+\varepsilon}\to\mathcal M^{c+\varepsilon}$ be the $C^1$-deformation given by Lemma 2.4. Then $\eta(1,\,\cdot\,)$ is a continuous and $\sigma$-equivariant map from $\mathcal M^{c+\varepsilon}\setminus\mathcal N$ to $\mathcal M^{c-\varepsilon}$. Using Lemma 3.1(c), we have $\gamma(\mathcal M^{c+\varepsilon}\setminus\mathcal N)\le\gamma(\mathcal M^{c-\varepsilon})$, and therefore

$$\gamma(\mathcal M^{c+\varepsilon})\le\gamma(\mathcal M^{c+\varepsilon}\setminus\mathcal N)+\gamma(\overline{\mathcal N})\le\gamma(\mathcal M^{c-\varepsilon})+\gamma(K_c).$$ □

Lemma 4.5.

(a) *For every $m$, $c_m<\infty$ is bounded independent of $\beta<0$.*
(b) *$c_m\to c^*$ as $m\to\infty$, where $\min_{0\le b\le a}\{c^{q_b}(\beta)\}\le c^*\le\infty$.*
(c) *If $c:=c_m=c_{m+1}=\ldots=c_l<\min_{0\le b\le a}\{c^{q_b}(\beta)\}$ for some $l\ge m$, then $\gamma(K_c)\ge l-m+1$.*
(d) *If $c_m<\min_{0\le b\le a}\{c^{q_b}(\beta)\}$, then $K_{c_m}\ne\emptyset$, and $\mathcal M^{c_m}$ contains at least $m$ $Z_N$ orbits of critical points of $E$.*

(e) *If $\beta \leq -1/(N_b-1)$, then there is no fixed point of $\sigma^{q_d}$ on $\mathcal{M}$ for $0 \leq d \leq b$.*

Proof. (a) By Proposition 4.1 or 4.2, there exists continuous map $\psi\colon \mathbb{S}^{2m-1} \to \mathcal{M}$ satisfying (3.1), i.e.

$$\psi\bigg(e^{-i2\pi/N}\sum_{k=1}^{m} r_k e^{i\theta_k}U^k(\rho,t)\bigg) = \sigma\psi\bigg(\sum_{k=1}^{m} r_k e^{i\theta_k}U^k(\rho,t)\bigg).$$

Then Lemma 3.2(f) implies $\gamma(\psi(\mathbb{S}^{2m-1})) \geq m$, and therefore

$$c_m \leq \sup_{U\in\mathbb{S}^{2m-1}} E(\psi(U)) < \infty.$$

By the definition of $\psi$ and the construction of $\mathbb{S}^{2m-1}$, the value of

$$\sup_{U\in\mathbb{S}^{2m-1}} E(\psi(U))$$

does not depend on $\beta$. Hence (a) follows.

(b) If the conclusion is not true, it must hold that $c^* < \min_{0\leq b\leq a}\{c^{q_b}(\beta)\}$ such that $c_m \to c^*$ as $m \to \infty$ since $\{c_m\}$ is a monotone increasing sequence. With similar argument as the proof of Lemma 4.4, there exists $\varepsilon > 0$ corresponding to $c^*$, such that

$$\gamma(\mathcal{M}^{c^*+\varepsilon}) \leq \gamma(\mathcal{M}^{c^*-\varepsilon}) + \gamma(K_{c^*}).$$

Choosing $m$ large such that $c_m > c^*-\varepsilon$, then the above inequality and Lemma 4.4 imply $\gamma(\mathcal{M}^{c^*+\varepsilon}) < \infty$. Now we choose $m' > \gamma(\mathcal{M}^{c^*+\varepsilon})$ and then the corresponding $c_{m'} \geq c^* + \varepsilon$, which is a contradiction. Thus $\min_{0\leq b\leq a}\{c^{q_b}(\beta)\} \leq c^* \leq \infty$.

(c) By Definition 3.1,

$$\gamma(\mathcal{M}^{c-\varepsilon}) \leq m-1 \quad \text{and} \quad \gamma(\mathcal{M}^{c+\varepsilon}) \geq l \quad \text{for all } \varepsilon > 0.$$

Then $\gamma(K_c) \geq l-m+1$ follows from Lemma 4.4.

(d) If $c_m < \min_{0\leq b\leq a}\{c^{q_b}(\beta)\}$, then we get $\gamma(K_{c_m}) \geq 1$ by choosing $l = m$ in (iii). Hence $K_{c_m}$ is not empty. If $c_1 < \ldots < c_m$, then $\mathcal{M}^{c_m}$ contains at least $m$ $Z_N$-orbits of critical points of $E$. If $c_i = c_j$ for some $i < j \leq m$, then $\gamma(K_{c_i}) > 1$. By Lemma 3.2(d), $K_{c_i}$ is an infinite set. Hence in either case we have at least $m$ $Z_N$-orbits of critical points of $E$.

(e) If $\beta \leq -1/(N_b-1)$ and $\overrightarrow{U}$ is fixed by $\sigma^{q_d}$, then

$$\begin{aligned}\|U_j\|^2 &= (1+\beta(N_d-1))|U_j^+|_4^4 + \beta(N_d-1)|U_j^-|_4^4 + \beta N_d\int\sum_{k\neq j}^{q_d} U_k^2U_j^2\\ &\leq (1+\beta(N_d-1))|U_j^+|_4^4 + \beta(N_d-1)|U_j^-|_4^4 \leq 0\end{aligned}$$

for $0 \leq d \leq b$. Thus $U_j = 0$ for all $1 \leq j \leq N$, which implies $\overrightarrow{U} = \overrightarrow{0}$. Note $\overrightarrow{0} \notin \mathcal{M}$, so we see that there is no fixed point of $\sigma^{q_d}$ on $\mathcal{M}$. Therefore (e) follows. $\square$

Proof of Theorem 1.1. (a) If $N$ is a prime number, then $c^{q_0}(\beta) = \infty$ for $\beta \leq -1/(N-1)$ by Lemma 2.3. Using Lemma 4.5(a) and (b), we choose $\overrightarrow{U}^m \in K_{c_m}$ for every positive integer $m$, then there exists a sequence of nontrivial $Z_N$-orbits of critical points of $E$. By Lemma 2.1, (1.3) has a sequence of nontrivial $Z_N$-orbits of solutions.

If $N$ is not prime, we have $N = q_b N_b$ for any fixed $b$, $1 \leq b \leq a$. Consider solutions of the form

$$(U_1, \ldots, U_{q_b}, U_1, \ldots, U_{q_b}, \ldots, U_1, \ldots, U_{q_b}), \tag{4.6}$$

which are fixed points of $\sigma^{q_b}$. In this case, the system (1.3) reduces to a system of $q_b$ equations

$$\begin{cases} -\Delta U_j + U_j = [1 + \beta(N_b - 1)]U_j^3 + \beta N_b \displaystyle\sum_{k \neq j} U_j U_k^2 & \text{in } \Omega, \\ U_j > 0 \quad \text{in } \Omega, \quad U_j = 0 \quad \text{on } \partial\Omega, \ j = 1, \ldots, q_b. \end{cases} \tag{4.7}$$

When $\beta > -1/(N_b - 1)$, system (4.7) is system (1.3) with $\widetilde{N} = q_b$, $\widetilde{\mu} = 1 + \beta(N_b - 1)$ and $\widetilde{\beta} = \beta N_b$. If, in some interval of $\beta$, system (4.7) has infinite sequence of $Z_{q_b}$-orbits of solutions, then according to (4.6), we get a infinite sequence of $Z_N$-orbits of solutions of system (1.3).

Define $b^* \in \mathbb{N}$ by $b^* = 0$ for $b = 1$, and for $b \geq 2$ by

$$b^* = \begin{cases} \min\{d \mid (q_e, q_b) = 1, \text{ for all } d < e < b\} & \text{if } (q_{b-1}, q_b) = 1, \\ b - 1 & \text{if } (q_{b-1}, q_b) > 1, \end{cases}$$

where $(q_e, q_b)$ denotes the greatest common divisor of $q_e$ and $q_b$.

By Lemma 4.5(e), there is no fixed point of $\sigma^{q_d}$ on $\mathcal{M}$, for any $0 \leq d \leq b^*$ when $\beta \leq -1/(N_{b^*} - 1)$. Consequently, there is no fixed point of $\sigma^{\widetilde{q}_e}$ on the corresponding Nehari manifold of (4.7) for $0 \leq e \leq f$, where $1 = \widetilde{q}_0 < \widetilde{q}_1 < \ldots < \widetilde{q}_f < q_b$ are distinct factors of $q_b$. Applying Lemma 2.3 to (4.7), we get $\min_{0 \leq e \leq f}\{c^{\widetilde{q}_e}(\beta)\} = \infty$ for $-1/(N_b - 1) < \beta \leq -1/(N_{b^*} - 1)$. Also, we use Lemma 4.5(a) and (b) to the corresponding critical levels of system (4.7). For every positive integer $m$, there exists a sequence of nontrivial $Z_{q_b}$-orbits of critical points of corresponding energy functional $\widetilde{E}$. By Lemma 2.1, (4.7) has a sequence of nontrivial $Z_{q_b}$-orbits of solutions. As a consequence, (1.3) admits an infinite sequence of $Z_N$-orbits of solutions.

Let $b$ run through 1 to $a$, we see that system (1.3) has a sequence of nontrivial $Z_N$-orbits of solutions for

$$\beta \in \bigcup_{b=1}^{a} \bigg( -\frac{1}{N_b - 1}, -\frac{1}{N_{b^*} - 1} \bigg] = \bigg( -\frac{1}{N_a - 1}, -\frac{1}{N - 1} \bigg].$$

Finally, if $\beta \le -1/(N_a - 1)$, by Lemma 2.3, we have $\min_{0 \le b \le a}\{c^{q_b}(\beta)\} = \infty$. Therefore, there is no $\sigma^{q_b}$ fixed point on $\mathcal{M}$ for all $b \le a$, or equivalently, for $\beta \le -1/(N_a - 1)$. Using Lemma 4.5, system (1.3) has an infinite sequence of nontrivial $Z_N$-orbits of solutions for $\beta \le -1/(N_a - 1)$. Therefore (a) holds.

(b) Let $m$ be a given positive integer. By Lemma 2.3 and Lemma 4.5(a), there exists $\beta_m > -1/(N - 1)$ such that for $\beta < \beta_m$ we have $c_m < \min_{0 \le b \le a}\{c^{q_b}(\beta)\}$. Hence $E$ has at least $m$ nontrivial critical points by Lemma 4.5(d) for $\beta < \beta_m$, and therefore problem (1.3) admits at least $m$ $Z_N$-orbits of solutions. □

Remark 4.6. When $\beta > -\mu/(N_b - 1)$, due to the invariant feature of system (1.3), the $Z_N$-orbits of solutions could be invariant under the action of $\sigma^{q_d}$ for some $0 \le d \le b$, where $0 \le b \le a$. But for $\beta \le -\mu/(N_b - 1)$, system (1.3) does not have $\sigma^{q_b}$ invariant solution any more. In particular, when $\beta > -\mu/(N_0 - 1) = -\mu/(N - 1)$, (1.3) may have solutions which are fixed points of $\sigma^{q_b}$ for $0 \le b \le a$; when $\beta \le -\mu/(N_a - 1)$, no $Z_N$-orbit of solution exists in the fixed point set of any $\sigma^b$, $0 \le b \le b \le N - 1$.

Remark 4.7. Our methods can be used to study a more general version of system (1.3)

$$\begin{cases} -\Delta U_j + U_j = \mu |U_j|^{2p-2} U_j + \beta \sum_{k \ne j} |U_k|^p |U_j|^{p-2} U_j & \text{in } \Omega, \\ U_j > 0 \quad \text{in } \Omega, \quad U_j = 0 \quad \text{on } \partial\Omega, \ j = 1, \dots, N \end{cases} \tag{4.8}$$

where $\mu > 0$ a constant, $N \ge 2$, $\Omega \subset \mathbb{R}^n$ is smooth bounded domain for $n \ge 1$, or radially symmetric (possibly unbounded) domain for $n \ge 2$, and

$$1 < p < \frac{2^*}{2} = \begin{cases} n/(n-2) & \text{for } n \ge 3, \\ \infty & \text{for } n = 1, 2. \end{cases}$$

With obvious changes (of notations, essentially) of the proof of Theorem 1.1, we can prove the following theorem. The details are omitted.

Theorem 4.8.

(a) *If $\beta \le -\mu/(N - 1)$, then system* (4.8) *has an infinite sequence of $Z_N$-orbits of solutions.*
(b) *For any positive integer $m$, there exists a $\beta_m \in (-\mu/(N - 1), 0)$, such that for $\beta \in (-\mu/(N - 1), \beta_m)$, system* (4.8) *has at least $m$ $Z_N$-orbits of solutions.*

**Acknowledgments.** The authors would like to thank the referee for carefully reading the manuscript and valuable comments. The second author is grateful to E. N. Dancer for a helpful discussion on related topics.

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*Manuscript received May 5, 2010*

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