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Standing Waves with a Critical Frequency for Nonlinear Schrödinger Equations

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Abstract

This paper is concerned with the existence and qualitative property of standing wave solutions $\psi(t,x) = e^{-iEt/\hbar}v(x)$ for the nonlinear Schrödinger equation $\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + |\psi|^{p-1}\psi = 0$ with E being a critical frequency in the sense that $\min_{\mathbb{R}^N} V(x) = E$. We show that there exists a standing wave which is trapped in a neighbourhood of isolated minimum points of V and whose amplitude goes to 0 as $\hbar \to 0$. Moreover, depending upon the local behaviour of the potential function $V(x)$ near the minimum points, the limiting profile of the standing-wave solutions will be shown to exhibit quite different characteristic features. This is in striking contrast with the non-critical frequency case ($\inf_{\mathbb{R}^N} V(x) > E$) which has been extensively studied in recent years.

1. Introduction

The evolution of a free non-relativistic quantum particle is described by linear Schrödinger equations, and this is one of the main results in quantum mechanics. On the other hand, for a group of identical particles interacting with each other in ultra-cold states, in particular, Bose-Einstein condensates, their evolution is described, via Hartree approximation, to an excellent degree of accuracy by nonlinear Schrödinger equations (see [Me]). The equation arises in many fields of physics, in particular, when we describe the propagation of light in some nonlinear optical materials; the nonlinear Schrödinger equations in nonlinear optics are reduced from Maxwell's equations (see [Mi]). The nonlinear Schrödinger equation is typically of the form

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + |\psi|^{p-1}\psi = 0, \tag{1}$$

where $\hbar$ denotes the Plank constant, i is the imaginary unit. In physical problems, a cubic nonlinearity, $p = 3$, is common; in this case, the equation is called the

Gross-Pitaevskii equation. In this paper we are concerned with the existence of standing waves of the nonlinear Schrödinger equation (1) for small $\hbar$. For small $\hbar > 0$, these standing-wave solutions are referred to as semi-classical states. Here a solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. Then, a function $\psi(x,t) \equiv \exp(-iEt/\hbar)\ v(x)$ is a standing-wave solution of (1) if and only if the function v satisfies

$$\tfrac{1}{2}\hbar^2\Delta v - (V(x) - E)v + |v|^{p-1}v = 0, \quad x \in \mathbb{R}^N. \tag{2}$$

If for some $\xi \in \mathbb{R}^N \setminus \{0\}$, $V(x + s\xi) = V(x), s \in \mathbb{R}$, equation (1) is invariant under a Galilean transformation,

$$\psi(x,t) \to \psi(x - \xi t, t)\exp(i\xi \cdot x/\hbar - \tfrac{1}{2}i|\xi|^2 t/\hbar)\psi(x - \xi t, t).$$

Thus, in this case, standing waves reproduce solitary waves travelling in the direction of ξ. In this paper, we investigate problem (2) when

$$\inf_{x\in\mathbb{R}^N} V(x) = E.$$

Under the condition that

$$\inf_{x\in\mathbb{R}^N} V(x) > E,$$

problem (2) has a ground-state solution (mountain-pass solution) for $\hbar > 0$ small (see [R]) when $\inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x)$; furthermore, the following well-understood problem in $\mathbb{R}^N$ plays a crucial role in the construction of solutions of (2) and is considered the limiting equation for (2) as $\hbar \to 0$:

$$\Delta u - u + |u|^{p-1}u = 0.$$

This problem has a unique ground-state solution $w > 0$:

On the other hand, for

$$\inf_{x\in\mathbb{R}^N} V(x) < E,$$

we can show easily that there are no ground-state solutions (mountain-pass solutions) of problem (2) if $\hbar > 0$ is sufficiently small; moreover, there is no nice limiting problem as in the case $E < \inf_{x\in\mathbb{R}^N} V(x)$. In this sense, we can say that $E = \inf_{x\in\mathbb{R}^N} V(x)$ is a critical frequency (or energy) for the nonlinear Schrödinger equation (1) or problem (2).

There have been enormous investigations on problem (2) under the condition

$$\inf_{x\in\mathbb{R}^N} V(x) > E.$$

FLOER & WEINSTEIN proved in [FW] that, for sufficiently small $\hbar > 0$, there exists a solution $u_\hbar$ of (2) with $\liminf_{\hbar\to 0} \max_{x\in\mathbb{R}^N} |u_\hbar(x)| > 0$ which is concentrated around a non-degenerate critical point of V when $N = 1$, $E < \inf_{\mathbb{R}^N} V(x)$, $p = 3$ and V is a bounded function. They used a Lyapunov-Schmidt reduction method to obtain the result. Further investigations and developments have been carried out by OH [O1–O3], WANG [W], RABINOWITZ [R], DEL PINO & FELMER [DF1–DF3], AMBROSETTI, BADIALE & CINGOLANI [ABC], GUI [G], LI [L], DANCER & YAN [DY], KANG & WEI [KW] and many others. (See also [By4], [Wa] and [CNY] for a radial potential V.)

In all the above-mentioned works, the authors use the ground-state solution w of the limiting equation stated above as a building block to construct single-bump or multi-bump solutions for (2) with each bump looking like a translated ground-state solution w. This ground-state solution w enjoys many nice properties which have been used in an essential way in the above works. Especially, the Lyapunov-Schmidt reduction method relies upon the uniqueness and nondegeneracy property of w. The limiting profiles of the perturbed equations are essentially determined by the ground-state solution w of the limiting equation.

In contrast, in our analysis of this paper we shall see that for the case of $E = \inf_{x\in\mathbb{R}^N} V(x)$ the situation changes dramatically and depends upon the local behaviour of V near its global minimum where $V = E$. Moreover, the limiting equations have many different forms, of which some are defined on R^N with homogeneous potentials and some are defined on bounded domains which could be of arbitrary shapes. These new features of the limiting equations provide new phenomena for the limiting profiles of the solutions of (2). For example, for all solutions obtained so far under condition $\inf_{R^N} V(x) > E$, the maximum values of the solutions are bounded away from zero (in fact, asymptotically they are proportional to the maximum value of w). On the other hand, as we will see in this paper, in the case of $E = \min_{x\in\mathbb{R}^N} V(x)$, there is a solution of (2) whose maximum value goes to 0 as $\hbar \to 0$; in nonlinear optics, this implies an existence of a standing light with very small intensity which is trapped in a neighbourhood of stable points of potential V. In fact, if A is an isolated component of $\{x \in \mathbb{R}^N | V(x) = E\}$, we find a solution $v_\hbar$ of (2) such that $\lim_{\hbar\to 0} \|v_\hbar\|_{L^\infty(\mathbb{R}^N)} = 0$, $\liminf_{\hbar\to 0} \hbar^{-2/(p-1)} \|v_\hbar\|_{L^\infty(\mathbb{R}^N)} > 0$ and $v_\hbar$ is exponentially small on $\mathbb{R}^N \setminus U$ as $\hbar \to 0$, where $\overline{A} \subset U$. This is in striking contrast with the case $\min_{x\in\mathbb{R}^N} V(x) > E$. Another new phenomenon that is highly contrary to that for the case $\min_{x\in\mathbb{R}^N} V(x) > E$ is observed when V is exponentially flat near an isolated zero. For example, for a potential V which decays exponentially around an isolated zero, there exists a least-energy solution of (2) in the class of even functions which has at least two local maximum points $h^1_\hbar, h^2_\hbar$ satisfying $\lim_{\hbar\to 0} |h^1_\hbar - h^2_\hbar| = 0$ (see Remark 3.3.3). It is well known that such kinds of phenomena do not occur around a local minimum point if $\min_{x\in\mathbb{R}^N} V(x) > E$.

We should note that we are more interested in the asymptotic behaviour of the least-energy solution while it is not difficult to show the existence of a least-energy solution even in the case $\min_{x\in\mathbb{R}^N} V(x) = E$. On the other hand, the existence of locally minimal energy solutions is rather difficult to demonstrate since the required solutions are very small and the energy of the solutions can be comparatively small with a very small perturbation of the required solution.

This paper is organized as follows. In Section 2, we prove the existence of localized solutions. Their asymptotic behaviour will be investigated in Section 3. Finally, we discuss some more interesting problems in Section 4.

2. Existence of localized solutions

We first rewrite (2) in the following form:

$$\begin{aligned} &\varepsilon^2\Delta v - V(x)v + v^p = 0,\ \ v > 0, \qquad \text{in} \quad \mathbb{R}^N \\ &\lim_{|x|\to 0} v(x) = 0, \end{aligned} \tag{3}$$

where $p \in (1, (N+2)/(N-2))$ for $N \geqq 3$, and $p \in (1, \infty)$ for $N = 1, 2$. We assume that the potential V satisfies the following conditions:

(V1) V is a continuous non-negative function on $\mathbb{R}^N$;
(V2) for some $\gamma > 0$, $\liminf_{|x|\to\infty} V(x) > 2\gamma$;
(V3) the zero set of V, $\mathcal{Z} \equiv \{x \in \mathbb{R}^N | V(x) = 0\}$ is non-empty.

Let A be an isolated component of $\mathcal{Z}$. We define $A^\delta = \{x \in \mathbb{R}^N | \operatorname{dist}(x, A) \leqq \delta\}$, and $A^\delta_\varepsilon = \{x \in \mathbb{R}^N | \varepsilon x \in A^\delta\}$. We assume that

(A) for some $\delta > 0$, $A^{8\delta} \cap (\mathcal{Z} \setminus A) = \emptyset$.

We define an energy functional

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + Vu^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx.$$

Positive critical points of $\Gamma_\varepsilon(u)$ are solutions of (3). This can also be rephrased as a minimization problem for the energy functional

$$I^\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + Vu^2\, dx$$

under the constraint $\int_{\mathbb{R}^N} |u|^{p+1} dx = 1$. A solution of (3) is called a least-energy solution if it minimizes I^ε.

Now we state the main result of this section on the existence of localized solutions.

Theorem 2.1. *Suppose that* (V1)–(V3) *and* (A) *hold. Then for sufficiently small* $\varepsilon > 0$, *there exists a solution* v_ε *of* (3) *such that*

(i) $\lim_{\varepsilon\to 0} \varepsilon^{-N}\Gamma_\varepsilon(v_\varepsilon) = 0$;
(ii) $\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0, \quad \liminf_{\varepsilon\to 0} \varepsilon^{-2/(p-1)}\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0$; *and*
(iii) *for each* $\delta' > 0$, *there exist constants* $C, c > 0$ *such that*

$$v_\varepsilon(x) \leqq C \exp\left(-\frac{c}{\varepsilon}\operatorname{dist}(x, A^{\delta'})\right).$$

On the least-energy solutions for (3), we have the following result similar to Theorem 2.1.

Theorem 2.2. *Suppose that* (V1)–(V3) *hold. Then, for sufficiently small* $\varepsilon > 0$, *there exists a least energy solution* v_ε *of* (3) *which satisfies*

(i) $\lim_{\varepsilon\to 0} \varepsilon^{-N}\Gamma_\varepsilon(v_\varepsilon) = 0$;
(ii) $\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0, \quad \liminf_{\varepsilon\to 0} \varepsilon^{-2/(p-1)}\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0$; *and*

(iii) *for each* $\delta' > 0$, *there exist constants* $C, c > 0$ *such that*

$$v_\varepsilon(x) \leqq C \exp\left(-\frac{c}{\varepsilon}\mathrm{dist}\left(x, \mathcal{Z}^{\delta'}\right)\right).$$

In what follows, we will proceed to prove Theorem 2.1. For the proof of Theorem 2.2, it suffices to take $A = \mathcal{Z}$ in Theorem 2.1.

By a rescaling, (3) is transformed to

$$\begin{aligned} &\Delta u - V(\varepsilon x)u + u^p = 0,\ u > 0, \qquad \text{in} \quad \mathbb{R}^N, \\ &\lim_{|x|\to 0} u(x) = 0. \end{aligned} \tag{4}$$

Define $V_\varepsilon(x) = V(\varepsilon x)$, $x \in \mathbb{R}^N$. We define a norm

$$\|u\|_\varepsilon \equiv \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V_\varepsilon(x)u^2\,dx\right)^{1/2},$$

and space H_ε as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_\varepsilon$.

We consider the following minimization problem:

$$\begin{aligned} I_A^\varepsilon \equiv \inf\Bigg\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V_\varepsilon u^2\,dx \Bigg| \int_{\mathbb{R}^N} u^{p+1}\,dx = 1, \\ \int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} u^{p+1}\,dx \leqq \varepsilon^{3(p+1)/(p-1)}, u \in H_\varepsilon \Bigg\}. \end{aligned}$$

Let us outline our proofs below. We will show that there exists a minimizer u_ε of I_A^ε satisfying

$$\int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} (u_\varepsilon)^{p+1}\,dx < \varepsilon^{3(p+1)/(p-1)}.$$

Then, $v_\varepsilon \equiv (I_A^\varepsilon)^{1/(p-1)}u_\varepsilon$ is a solution of (3), and v_ε satisfies the properties (i)–(iii) of Theorem 2.1. The first step is to prove the existence of a minimizer of I_A^ε in the following Proposition 2.4. The second step is to estimate the minimizer on a neighbourhood of $\mathcal{Z}\setminus A$ and out of $\mathcal{Z}$; this will be done in Lemmas 2.7 and 2.8. The last step is to prove the properties (i)–(iii), and this will complete the proof of Theorem 2.1.

First we have

Lemma 2.3. *The following equality holds:*

$$\lim_{\varepsilon\to 0} I_A^\varepsilon = 0.$$

Proof. Let $x_0 \in A$. Then, for any $a > 0$, there exists $b \in (0, \delta)$ such that $V(x) \in [0, a)$ for $|x - x_0| \leqq b$. Then, we see that

$$\begin{aligned} I_A^\varepsilon \leqq I_a^\varepsilon \equiv \inf\Bigg\{ \int_{B(x_0,b/\varepsilon)} |\nabla u|^2 + au^2\,dx \Bigg| \int_{B(x_0,b/\varepsilon)} u^{p+1}\,dx = 1, \\ u \in C_0^\infty(B(x_0, b/\varepsilon)) \Bigg\}. \end{aligned}$$

It is obvious that

$$\lim_{\varepsilon\to 0} I_a^\varepsilon = I_a \equiv \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + au^2\,dx \,\middle|\, \int_{\mathbb{R}^N} u^{p+1}\,dx = 1, u \in H^{1,2}(\mathbb{R}^N)\right\}.$$

Through a simple calculation, we see that

$$I_a = a^{1-\frac{n}{2}\frac{p+1}{p-1}} I_1\,.$$

Therefore, since $1-\frac{n}{2}\frac{p+1}{p-1} > 0$, it follows that $\lim_{\varepsilon\to 0} I_A^\varepsilon = 0$. □

Proposition 2.4. *For sufficiently small $\varepsilon > 0$, I_A^ε is attained by some $u_\varepsilon \in H_\varepsilon$.*

Proof. Let $\{v_\varepsilon^n\}_{n=1}^\infty \subset H_\varepsilon$ be a minimizing sequence for I_A^ε. We can assume $\{v_\varepsilon^n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ and that v_ε^n converges weakly to some v_ε in H_ε as $n\to\infty$. Since, in general, an imbedding $H_\varepsilon \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is not compact, it does not follow always that $\int_{\mathbb{R}^N} v_\varepsilon^{p+1}\,dx = 1$ and $\int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} v_\varepsilon^{p+1}\,dx \leqq \varepsilon^{3(p+1)/(p-1)}$. To overcome this difficulty, we first find a refined minimizing sequence $\{u_\varepsilon^n\}_{n=1}^\infty$ as follows, which possesses a nicer property.

We take $R_n > 0$ such that $\operatorname{supp}(v_\varepsilon^n) \subset B(0,R_n) \equiv \{x \in \mathbb{R}^N \mid |x| \leqq R_n\}$. We can assume that $R_1 \leqq R_2 \leqq R_3 \ldots$, $\mathcal{Z}_\varepsilon^{4\delta} \subset B(0,R_1)$ and $\lim_{n\to\infty} R_n = \infty$. We define the completion of $C_0^\infty(B(0,R_n))$ with respect to the norm $\|\cdot\|_\varepsilon$ by H_ε^n. Then, we consider the following minimization problem:

$$I_{A,n}^\varepsilon \equiv \inf\left\{\int_{B(0,R_n)} |\nabla u|^2 + V_\varepsilon u^2\,dx \,\middle|\, \int_{B(0,R_n)} u^{p+1}\,dx = 1,\right.$$
$$\left.\int_{B(0,R_n)\setminus A_\varepsilon^{4\delta}} u^{p+1}\,dx \leqq \varepsilon^{3(p+1)/(p-1)}, u \in H_\varepsilon^n\right\}.$$

It is easy to check that $\{I_{A,n}^\varepsilon\}_{\varepsilon,n}$ is bounded. Since H_ε^n is compactly imbedded in $L^{p+1}(B(0,R_n))$, it is easy to see that there exists a non-negative minimizer u_ε^n of $I_{A,n}^\varepsilon$. Note that for any $k \geqq 1$,

$$\lim_{n\to\infty}\int_{B(0,R_n)} |\nabla u_\varepsilon^n|^2 + V_\varepsilon (u_\varepsilon^n)^2\,dx \leqq \int_{\mathbb{R}^N} |\nabla v_\varepsilon^k|^2 + V_\varepsilon (v_\varepsilon^k)^2\,dx.$$

Thus, $\{u_\varepsilon^n\}_{n=1}^\infty$ is a minimizing sequence for I_A^ε.

Since u_ε^n is a minimizer for $I_{A,n}^\varepsilon$, there exist Lagrange multipliers $\alpha_\varepsilon(n), \beta_\varepsilon(n) \in \mathbb{R}$ such that

$$\begin{aligned} \Delta u_\varepsilon^n - V_\varepsilon u_\varepsilon^n + \alpha_\varepsilon(n)(u_\varepsilon^n)^p + \beta_\varepsilon(n)\chi_{B(0,R_n)\setminus A_\varepsilon^{4\delta}}(u_\varepsilon^n)^p &= 0 && \text{in } B(0,R_n),\\ u_\varepsilon^n &> 0 && \text{in } B(0,R_n),\\ u_\varepsilon^n &= 0 && \text{on } \partial B(0,R_n), \end{aligned}$$

where a characteristic function χ_B is defined by $\chi_B(x) = 1$ for $x \in B$, $\chi_B(x) = 0$ for $x \notin B$. As in [By1], we can show that $\beta_\varepsilon(n) \leqq 0 \leqq \alpha_\varepsilon(n)$.

We claim that $\{\alpha_\varepsilon(n)\}_n$ is uniformly bounded for small $\varepsilon > 0$, i.e., there exist $\varepsilon_0 > 0$ and $M > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$, $\alpha_\varepsilon(n) \leqq M$ for all n.

On the contrary, taking a subsequence of $\{R_n\}_n$ if it is necessary, we assume that there exist positive numbers $\{\varepsilon_n\}_n$ such that $\lim_{n\to\infty}\varepsilon_n = 0$ and $\lim_{n\to\infty}\alpha_{\varepsilon_n}(n) = \infty$. For the sake of convenience, we denote $\alpha_n = \alpha_{\varepsilon_n}(n)$ and $u^n = u^n_{\varepsilon_n}$. We take $\varphi_m \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_m(x) = \begin{cases} 1 \text{ if } \operatorname{dist}(x, \mathbb{R}^N \setminus A^{4\delta}_{\varepsilon_n}) \geqq 1/m, \\ 0 \text{ if } x \notin A^{4\delta}_{\varepsilon_n}, \end{cases}$$

and such that $0 \leqq \varphi_m \leqq 1$, $|\nabla\varphi_m| \leqq 2m$. Then, multiplying both sides of the above equation by φ_m and integrating by parts, we see that

$$\begin{aligned} &\alpha_n \int_{\{x\in\mathbb{R}^N | \operatorname{dist}(x,\mathbb{R}^N\setminus A^{4\delta}_{\varepsilon_n})\geqq 1/m\}} (u^n)^{p+1}\,dx \\ &\leqq \int_{\mathbb{R}^N} |\nabla u^n|^2 + |\nabla u^n||\nabla\varphi_m|u^n + V_{\varepsilon_n}(u^n)^2\,dx. \end{aligned}$$

Note that $\inf_{x\in A^{4\delta}\setminus A^{3\delta}} V(x) > 0$. Then, from Cauchy's inequality, it follows that for each $m \geqq 1$, there is $C > 0$ satisfying

$$\alpha_n \int_{\{x\in\mathbb{R}^N | \operatorname{dist}(x,\mathbb{R}^N\setminus A^{4\delta}_{\varepsilon_n})\geqq 1/m\}} (u^n)^{p+1}\,dx \leqq C\int_{\mathbb{R}^N} |\nabla u^n|^2 + V_{\varepsilon_n}(u^n)^2\,dx$$

if n is sufficiently large. This implies that for each $m \geqq 1$,

$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}^N | \operatorname{dist}(x,\mathbb{R}^N\setminus A^{4\delta}_{\varepsilon_n})\geqq 1/m\}} (u^n)^{p+1}\,dx = 0.$$

We take $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi_\varepsilon(x) = \begin{cases} 0, & x \in A^{2\delta}_\varepsilon, \text{ or } x \notin A^{5\delta}_\varepsilon, \\ 1, & x \in A^{4\delta}_\varepsilon \setminus A^{3\delta}_\varepsilon, \end{cases}$$

$0 \leqq \psi_\varepsilon \leqq 1$, and $\|\nabla\psi_\varepsilon\|_{L^\infty} \leqq 4\varepsilon/\delta$. Then, we see that

$$\lim_{n\to\infty} \|u^n\psi_{\varepsilon_n}\|_{L^{p+1}} = 1,$$

and that $\{\|u^n\psi_{\varepsilon_n}\|_{\varepsilon_n}\}$ is bounded. We note that $\inf_{x\in A^{5\delta}\setminus A^{2\delta}} V(x) > 0$. This implies that $u^n\psi_{\varepsilon_n}$ is bounded in $H^1(R^N)$. Then, by the concentration-compactness lemma of Lions [Lio2, Lemma I.1] (see also [By1, Lemma 3.4]) there is $b > 0$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B(x,b)} (u^n)^{p+1}\,dx \geqq q \lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B(x,b)} (u^n\psi_{\varepsilon_n})^{p+1}\,dx > 0.$$

Thus, we can take $x_{\varepsilon_n} \in \partial A^{4\delta}_{\varepsilon_n}$ and $a \in (0, 1)$ such that

$$\liminf_{n\to\infty} \int_{B(x_{\varepsilon_n},b)} (u^n\psi_{\varepsilon_n})^{p+1}\,dx \geqq a.$$

We take $\psi_\varepsilon^d \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi_\varepsilon^d(x) = \begin{cases} 0, & \operatorname{dist}(x, \partial A_\varepsilon^{4\delta}) \geqq 2d, \\ 1, & \operatorname{dist}(x, \partial A_\varepsilon^{4\delta}) \leqq d, \end{cases}$$

$0 \leqq \psi_\varepsilon^d \leqq 1$, and $\|\nabla \psi_\varepsilon^d\|_{L^\infty} \leqq 4/d$. From the Poincaré inequality, we see that for some $D > 0$, independent of d and n,

$$\int_{B(x_{\varepsilon_n}, b)} (\psi_{\varepsilon_n}^d u^n)^2 \, dx \leqq Dd^2 \int_{B(x_{\varepsilon_n}, b)} |\nabla \psi_{\varepsilon_n}^d u^n|^2 \, dx.$$

Note that

$$\begin{aligned} &\int_{B(x_{\varepsilon_n}, b)} |\nabla \psi_{\varepsilon_n}^d u^n|^2 \, dx \\ &= \int_{B(x_{\varepsilon_n}, b)} |\nabla u^n|^2 (\psi_{\varepsilon_n}^d)^2 + \nabla \psi_{\varepsilon_n}^d \cdot \nabla u^n \psi_{\varepsilon_n}^d u^n + (u^n)^2 |\nabla \psi_{\varepsilon_n}^d|^2 \, dx \\ &\leqq 2 \int_{B(x_{\varepsilon_n}, b)} |\nabla u^n|^2 \, dx + (u^n)^2 |\nabla \psi_{\varepsilon_n}^d|^2 \, dx \\ &\leqq 2 \int_{B(x_{\varepsilon_n}, b)} |\nabla u^n|^2 \, dx \\ &\quad + \frac{8}{d^2} |B(x_{\varepsilon_n}, b)|^{\frac{p-1}{p+1}} \left(\int_{\{x \in B(x_{\varepsilon_n}, b) \mid \operatorname{dist}(x, \partial A_{\varepsilon_n}^{4\delta}) \geqq 2d\}} (u^n)^{p+1} \, dx \right)^{\frac{2}{p+1}}, \end{aligned}$$

and that for each $d > 0$,

$$\lim_{n \to \infty} \int_{\{x \in B(x_{\varepsilon_n}, b) \mid \operatorname{dist}(x, \partial A_{\varepsilon_n}^{4\delta}) \geqq 2d\}} (u^n)^{p+1} \, dx = 0.$$

Thus, it follows that

$$\lim_{n \to \infty} \int_{B(x_{\varepsilon_n}, b)} (u^n)^2 \, dx = 0.$$

From Hölder's inequality, the Sobolev inequality and the fact that the set $\{\|\psi_{\varepsilon_n} u^n\|_{\varepsilon_n}\}_n$ is bounded, we obtain

$$\lim_{n \to \infty} \int_{B(x_{\varepsilon_n}, b)} |\psi_{\varepsilon_n} u^n|^{p+1} \, dx = 0.$$

This is a contradiction.

Since a set $\{\alpha_\varepsilon(n)\}_n$ is uniformly bounded for small $\varepsilon > 0$ and $\int_{\mathbb{R}^N} (u_\varepsilon^n)^{p+1} \, dx = 1$, it follows that $\{\|u_\varepsilon^n\|_{L^\infty(\mathbb{R}^N)}\}_n$ is uniformly bounded for small $\varepsilon > 0$ (see [By2, Proposition 3.5]). From condition (V3), we can take $R = R(\varepsilon) > 0$ such that $V(x) \geqq \gamma$ for $|x| \geqq R/2$, and such that $A_\varepsilon^{4\delta} \subset B(0, R/2)$. Note that $\Delta u_\varepsilon^n - V_\varepsilon u_\varepsilon^n + \alpha_\varepsilon(n)(u_\varepsilon^n)^{p-1} u_\varepsilon^n \geqq 0 \quad \text{in } B(0, R_n)$, and that $\{\|\alpha_\varepsilon(n)(u_\varepsilon^n)^{p-1}\|_{L^\infty(\mathbb{R}^N)}\}_n$ is bounded. Then, since

$$\int_{\mathbb{R}^N \setminus A_\varepsilon^{4\delta}} (u_\varepsilon^n)^{p+1} \, dx \leqq \varepsilon^{3(p+1)/(p-1)},$$

we see from [GT, Theorem 9.26] that

$$\|u_\varepsilon^n\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))} \leqq C\varepsilon^{3/(p-1)},$$

where C is a positive constant, independent of n. Then, for sufficiently small $\varepsilon > 0$,

$$\|\alpha_\varepsilon(n)(u_\varepsilon^n)^{p-1}\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))} \leqq \gamma/2.$$

Then, from the maximum principle, we deduce that for some constant $C > 0$,

$$u_\varepsilon^n(x) \leqq C\exp(-\gamma(|x|-R)/4), \qquad |x| \geqq R. \tag{5}$$

We assume that u_ε^n converges weakly to some u_ε in H_ε as $n \to \infty$. Then, we know that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2\,dx \leqq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_\varepsilon^n|^2 + V_\varepsilon(u_\varepsilon^n)^2\,dx = I_A^\varepsilon.$$

From (5), we see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (u_\varepsilon^n)^{p+1}\,dx = \int_{\mathbb{R}^N} (u_\varepsilon)^{p+1}\,dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} (u_\varepsilon^n)^{p+1}\,dx = \int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} (u_\varepsilon)^{p+1}\,dx.$$

This implies that u_ε is a minimizer of I_A^ε. This completes the proof of Proposition 2.4. □

Since $u_\varepsilon \in H_\varepsilon$ is a non-negative minimizer of I_A^ε, there are Lagrange multipliers $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{R}$ such that

$$\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + \alpha(\varepsilon)u_\varepsilon^p + \beta(\varepsilon)\chi_\varepsilon u_\varepsilon^p = 0 \text{ in } \mathbb{R}^N,$$

where

$$\chi_\varepsilon(x) = \begin{cases} 0, & x \in A_\varepsilon^{4\delta}, \\ 1, & x \notin A_\varepsilon^{4\delta}. \end{cases}$$

By the same argument as in the proof of Proposition 2.4 (see also [By1]), we see that $\beta(\varepsilon) \leqq 0 \leqq \alpha(\varepsilon)$. By using similar arguments to these used in the proof of Proposition 2.4, we have the following lemma whose proof is omitted.

Lemma 2.5. *The set $\{\alpha(\varepsilon)\}_\varepsilon$ is bounded.*

We have the following decay property of u_ε.

Lemma 2.6. *The following equality holds:*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\setminus \mathcal{Z}_\varepsilon^{2\delta})} = 0.$$

Proof. We can show the boundedness of $\{\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\backslash\mathcal{Z}_\varepsilon^\delta)}\}_\varepsilon$ by bootstrap arguments or the Moser iteration technique (see [By2, Proposition 3.5] or [GT]). Note that $\inf\{V(x)|\ x\in\mathbb{R}^N\backslash\mathcal{Z}^\delta\}>0$. Thus, combining an elliptic estimate [GT, Theorem 9.20], the Sobolev inequality and Lemma 2.4, we see that for some $c, C>0$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\backslash\mathcal{Z}_\varepsilon^{2\delta})} &\leqq c\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^N\backslash\mathcal{Z}_\varepsilon^\delta}|\nabla u_\varepsilon|^2+u_\varepsilon^2\,dx\right)^{1/2}\\ &\leqq cC\|u_\varepsilon\|_\varepsilon=0.\end{aligned}$$

□

Then, we have the following exponential decay of u_ε.

Lemma 2.7. *For some $c, C>0$,*

$$u_\varepsilon(x)\leqq C\exp(-c\,\mathrm{dist}(x,\mathcal{Z}_\varepsilon^{2\delta})).$$

Proof. Let $3c=\inf\{V(x)|\ x\in\mathbb{R}^N\backslash\mathcal{Z}^{2\delta}\}>0$. Since $\beta(\varepsilon)\leqq 0\leqq\alpha(\varepsilon)$, from Lemmas 2.5 and 2.6, it follows that for sufficiently small $\varepsilon>0$,

$$\Delta u_\varepsilon(x)-(V_\varepsilon(x)-c)u_\varepsilon\geqq 0,\qquad x\in\mathbb{R}^N\backslash\mathcal{Z}_\varepsilon^{2\delta}.$$

We find a set $\mathcal{B}_\varepsilon$ containing $\mathcal{Z}_\varepsilon^{2\delta}$ such that $\partial\mathcal{B}_\varepsilon$ is smooth and $\max\{|x-y|\ |x\in\mathcal{B}_\varepsilon, y\in\mathcal{Z}_\varepsilon^{2\delta}\}\leqq 1$. We consider a problem

$$\begin{aligned}\Delta U-2cU&=0\qquad\text{in}\quad\mathbb{R}^N\backslash\mathcal{B}_\varepsilon,\\ U&=1\qquad\text{in}\quad\partial\mathcal{B}_\varepsilon,\\ \lim_{|x|\to\infty}U(x)&=0.\end{aligned}$$

Then, there exists a unique positive solution U of above equation such that for some $C'>0$,

$$U(x)\leqq C'\exp(-c\,\mathrm{dist}(x,\mathcal{B}_\varepsilon)),\qquad x\in\mathbb{R}^N\backslash\mathcal{B}_\varepsilon.$$

Thus, by the maximum principle [PW], we deduce that for some $C>0$,

$$u_\varepsilon(x)\leqq C\exp(-c\,\mathrm{dist}(x,\mathcal{Z}_\varepsilon^{2\delta})).$$

□

Since V is 0 on $\mathcal{Z}\backslash A$, we cannot apply directly maximum principles to get good decay estimates on a neighbourhood of $\mathcal{Z}\backslash A$. On the other hand, from the second constraint $\int_{\mathbb{R}^N\backslash A_\varepsilon^{4\delta}}u^{p+1}\,dx\leqq\varepsilon^{3(p+1)/(p-1)}$, we can obtain a good decay estimate comparing u_ε with the first eigenfunction on a neighbourhood of $\mathcal{Z}\backslash A$.

Lemma 2.8. *For some constant $C, c>0$,*

$$\|u_\varepsilon\|_{L^\infty(\mathcal{Z}_\varepsilon^{3\delta}\backslash A_\varepsilon^{4\delta})}\leqq C\exp(-c\delta/\varepsilon).$$

Proof. We find the first eigenfunction Φ and the first eigenvalue λ_1 of $-\Delta$ on $\mathrm{int}(\mathcal{Z}^{4\delta})$ with the homogeneous Dirichlet boundary condition. We can assume that $\Phi(x) \geqq 1$ for $x \in \mathcal{Z}^{3\delta}$, and that $M \equiv \max\{\Phi(x) \mid x \in \mathcal{Z}^{4\delta}\} < \infty$. For $a, b > 0$, we define $\Phi_\varepsilon(x) \equiv a\exp(-\frac{b}{\varepsilon})\Phi(\varepsilon x)$. Then, we see that

$$\Delta\Phi_\varepsilon + \varepsilon^2\lambda_1\Phi_\varepsilon = 0 \quad \text{in } \mathrm{int}(\mathcal{Z}_\varepsilon^{4\delta}),$$
$$\Phi_\varepsilon \geqq a\exp\left(-\frac{b}{\varepsilon}\right) \quad \text{on } \partial\left(\mathcal{Z}_\varepsilon^{3\delta}\right).$$

Since $\int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} u_\varepsilon^{p+1}\,dx \leqq \varepsilon^{3(p+1)/(p-1)}$ and $\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + \alpha(\varepsilon)u_\varepsilon^p \geqq 0$ in $\mathbb{R}^N$, we see from [GT, Theorem 9.26] that

$$\|u_\varepsilon\|_{L^\infty(\mathcal{Z}_\varepsilon^{4\delta}\setminus A_\varepsilon^{4\delta})} \leqq C\varepsilon^{3/(p-1)},$$

where C is a positive constant, independent of small $\varepsilon > 0$. Thus, for some constants $C_1, C_2, C_3 > 0$, it follows that

$$\Delta u_\varepsilon + C_1\varepsilon^3 u_\varepsilon \geqq 0 \quad \text{in } \mathrm{int}(\mathcal{Z}_\varepsilon^{4\delta}\setminus A_\varepsilon^{4\delta}),$$
$$u_\varepsilon \geqq C_1\exp\left(-\frac{C_2}{\varepsilon}\right) \quad \text{on } \partial(\mathcal{Z}_\varepsilon^{3\delta}\setminus A_\varepsilon^{4\delta}).$$

Therefore, by the comparison principles [PW], we deduce that for some constant $C, c > 0$,

$$\|u_\varepsilon\|_{L^\infty(\mathcal{Z}_\varepsilon^{3\delta}\setminus A_\varepsilon^{4\delta})} \leqq C\exp(-c\delta/\varepsilon).$$

□

Proof of Theorem 2.1. From Lemmas 2.7 and 2.8, we see that

$$\int_{\mathbb{R}^N\setminus A_\varepsilon^{4\delta}} u_\varepsilon^{p+1}\,dx < \varepsilon^{3(p+1)/(p-1)}.$$

That is, u_ε is a local minimizer of the constrained problem related to (4). Thus, defining $U_\varepsilon \equiv \alpha(\varepsilon)^{1/(p-1)}u_\varepsilon$, we have U_ε satisfying (4). Moreover, defining $v_\varepsilon(x) = U_\varepsilon(\varepsilon x)$, we see that v_ε satisfies (3). It is easy to see that $\alpha(\varepsilon) = I_A^\varepsilon$, and that $\Gamma(v_\varepsilon) = \varepsilon^{-N}(I_A^\varepsilon)^{(p+1)/(p-1)}$. Thus, Theorem 2.1(i) follows.

The first part of Theorem 2.1(ii) comes from the fact that

$$\lim_{\varepsilon\to 0}\|U_\varepsilon\|_\varepsilon = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(U_\varepsilon)^{p+1}\,dx = \lim_{\varepsilon\to 0}(I_A^\varepsilon)^{(p+1)/(p-1)} = 0$$

and [By2, Proposition 3.5] or [GT, Theorem 9.26].

To show the second part of Theorem 2.1(ii), i.e., $\liminf_{\varepsilon\to 0}\varepsilon^{-2/(p-1)}\|v_\varepsilon\|_{L^\infty} > 0$, let us define $w_\varepsilon \equiv \varepsilon^{-2/(p-1)}v_\varepsilon$. Then, it suffices to show $\liminf_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty} > 0$. It is easy to see

$$\Delta w_\varepsilon - \frac{1}{\varepsilon^2}Vw_\varepsilon + (w_\varepsilon)^p = 0 \qquad \text{in } \mathbb{R}^N.$$

Multiplying the above equation through by w_ε and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon^2} V(w_\varepsilon)^2\,dx \leqq \|(w_\varepsilon)^{p-1}\|_{L^\infty} \int_{\mathbb{R}^N} (w_\varepsilon)^2\,dx.$$

We take $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ for $x \in \mathcal{Z}^{4\delta}$. Then, we see that for some constant $C, c > 0$, independent of small $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (w_\varepsilon)^2\,dx \\
&\quad\leqq 2\int_{\mathbb{R}^N} \phi^2(w_\varepsilon)^2 + (1-\phi)^2(w_\varepsilon)^2\,dx \\
&\quad\leqq C\int_{\mathbb{R}^N} |\nabla(\phi w_\varepsilon)|^2\,dx + C\int_{\mathbb{R}^N} \frac{1}{\varepsilon^2} V(w_\varepsilon)^2\,dx \\
&\quad\leqq 2C\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + |\nabla\phi|^2(w_\varepsilon)^2\,dx + C\int_{\mathbb{R}^N} \frac{1}{\varepsilon^2} V(w_\varepsilon)^2\,dx \\
&\quad\leqq 2C\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + 2Cc\int_{\mathbb{R}^N} \frac{1}{\varepsilon^2} V(w_\varepsilon)^2\,dx + C\int_{\mathbb{R}^N} \frac{1}{\varepsilon^2} V(w_\varepsilon)^2\,dx. \quad (6)
\end{aligned}$$

This together with the previous inequality implies that for some constant $C > 0$, independent of $\varepsilon > 0$,

$$\|(w_\varepsilon)^{p-1}\|_{L^\infty} \geqq qC.$$

This proves Theorem 2.1(ii).

Theorem 2.1(iii) comes from Lemmas 2.4–2.8 and the comparison principles. □

3. Asymptotic profiles for localized solutions

As we have seen in Section 2, the localized solution v_ε has a small peak: its maximum value goes to 0 as $\varepsilon \to 0$. This is quite in contrast with the case where the potential V is bounded away from 0. In this section, we will investigate the asymptotic profile of localized solutions yielding more fine properties of these solutions. As we will prove, the asymptotic behaviour of localized solutions given in Theorem 2.1 depends in a very delicate way on some local properties of V around A. We distinguish three cases here: (i) The flat case, where the interior of A, $\mathrm{int}(A)$, is non-empty. (ii) The finite case, where A is a single point and V behaves like a finite-order polynomial near A. (iii) The infinite case, where A is a single point and V is exponentially flat near A. Though these three cases do not cover all possible local behaviours of V around A, they are the most typical models. Finer analysis is needed for more complicated cases.

To find asymptotic profiles for each case, it is essential to take appropriate normalizations so that the normalized problem has a non-trivial limiting problem. The normalization is closely related to a decay property of V around A.

Notation: we say that a family of functions $\{u_\varepsilon\}_\varepsilon$ subconverges in a space X as $\varepsilon \to 0$ if for each sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} \varepsilon_n = 0$, there exists a subsequence $\{\varepsilon_{n_i}\}_{i=1}^\infty$ of $\{\varepsilon_n\}_{n=1}^\infty$ such that $u_{\varepsilon_{n_i}}$ converges in X as $i \to \infty$.

3.1. The flat case

In this subsection, we consider a case in which the set of interior points of A, $\mathrm{int}(A)$, is not empty, and $A = \overline{\mathrm{int}(A)}$. Let $\mathrm{int}(A) = \cup_{i\in J} A_i$, where $\{A_i\}_{i\in J}$ are connected components of $\mathrm{int}(A)$. For each $i \in J$, we consider the following problem

$$\begin{aligned} \Delta u + u^p &= 0 \quad \text{in} \quad A_i, \\ u &> 0 \quad \text{in} \quad A_i, \\ u &= 0 \quad \text{on} \quad \partial A_i, \end{aligned} \tag{7-i}$$

Each problem (7-i) has a least-energy solution U_i. Now, let v_ε be a localized solution given in Theorem 2.1. We scale it as $v_\varepsilon(x) \equiv \varepsilon^{2/(p-1)} w_\varepsilon(x)$. Then, it follows that

$$\Delta w_\varepsilon - \frac{1}{\varepsilon^2} V(x) w_\varepsilon + w_\varepsilon^p = 0 \text{ in } \mathbb{R}^N.$$

Then, it is easy to see that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + \frac{V}{\varepsilon^2}(w_\varepsilon)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} (w_\varepsilon)^{p+1}\,dx \\ &\quad \leqq \inf_{i\in J}\left\{\frac{1}{2}\int_{A_i} |\nabla U_i|^2 dx - \frac{1}{p+1}\int_{A_i} (U_i)^{p+1}\,dx\right\}. \end{aligned} \tag{8}$$

Thus, from Theorem 2.1(iii), we deduce that set $\{\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + (w_\varepsilon)^2\,dx\}_\varepsilon$ is bounded. We can assume that for some $w \in H^{1,2}(\mathbb{R}^N)$, w_ε subconverges weakly in $H^{1,2}(\mathbb{R}^N)$ and pointwise to w as $\varepsilon \to 0$. Then, from Theorem 2.1(iii), we see that $w(x) = 0$ for $x \notin A$, and that w_ε subconverges to w in $L^{p+1}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Thus, for any test function $\varphi \in C_0^\infty(\mathrm{int}(A))$,

$$0 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \nabla w_\varepsilon \cdot \nabla\varphi + \frac{1}{\varepsilon^2} V w_\varepsilon \varphi - (w_\varepsilon)^p \varphi\,dx = \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi - w^p \varphi\,dx.$$

Thus, we see that

$$\begin{aligned} \Delta w + w^p &= 0 \quad \text{in} \quad \mathrm{int}(A), \\ w &\geqq 0 \quad \text{in} \quad A, \\ w &= 0 \quad \text{on} \quad \partial A. \end{aligned} \tag{9}$$

Since $\cup_{i\in J} A_i$ is bounded, it is rather easy to see that for some $C, c > 0$, independent of $i \in J$,

$$\frac{1}{2}\int_{A_i} |\nabla U_i|^2\,dx - \frac{1}{p+1}\int_{A_i} (U_i)^{p+1}\,dx \geqq C|A_i|^{-c},$$

where $|A_i|$ means the N-dimensional volume of A_i. Then, since $\sum_{i\in J} |A_i| < \infty$, it follows that there exist only finite members $j \in J$ such that

$$\begin{aligned}
&\frac{1}{2}\int_{A_j} |\nabla U_j|^2\,dx - \frac{1}{p+1}\int_{A_j} (U_j)^{p+1}\,dx \\
&\quad = \inf_{i\in J}\left\{\frac{1}{2}\int_{A_i} |\nabla U_i|^2\,dx - \frac{1}{p+1}\int_{A_i} (U_i)^{p+1}\,dx\right\} \\
&\quad \equiv F,
\end{aligned}$$

From Theorem 2.1(ii) and (8), we deduce that for some least-energy solution U_j of (7-j) satisfying $\frac{1}{2}\int_{A_j} |\nabla U_j|^2\,dx - \frac{1}{p+1}\int_{A_j}(U_j)^{p+1}\,dx = F$,

$$w = U_j \quad \text{on } A_j \qquad \text{and} \qquad w = 0 \quad \text{on } \operatorname{int}(A)\backslash A_j.$$

Moreover, from elliptic estimates, we see that for each compact set $D \subset \operatorname{int}(A)$, the convergence of w_ε is uniform on D. Thus, for each $\delta > 0$, w_ε subconverges uniformly to w on $\mathbb{R}^N \setminus (\partial A)^\delta$ as $\varepsilon \to 0$, where $(\partial A)^\delta \equiv \{x \in \mathbb{R}^N | \operatorname{dist}(x, \partial A)\} < \delta$. Therefore we obtain the following result.

Theorem 3.1. *The following equality holds:*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2(p+1)/(p-1)}\Gamma_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{A_j} |\nabla U_j|^2\,dx - \frac{1}{p+1}\int_{A_j} (U_j)^{p+1}\,dx.$$

Moreover, $\varepsilon^{2/(1-p)}v_\varepsilon$ subconverges pointwise to some least-energy solution U_j of (7-j) on A_j and to 0 *on $\mathbb{R}^N\backslash A_j$. For each $\delta > 0$, the convergence is uniform on* $\{x \in \mathbb{R}^N | \operatorname{dist}(x, \partial A) \geqq \delta\}$.

3.2. The finite case

If a non-negative potential V is analytic at 0 and $V(0) = 0$, then

$$V(x) = P_{2m}(x) + o(|x|^{2m}) \text{ as } |x| \to 0,$$

where P_{2m} is a homogeneous polynomial of order $2m$. In this case, as we will see later, a decay rate of $\|v_\varepsilon\|_{L^\infty}$ depends on $2m$ and P_{2m}. In this section, we will investigate asymptotic behaviour of v_ε for certain potential functions which are *asymptotically homogeneous* at zero.

Definition 3.2.1. For $m \in (0, \infty)$, a continuous function $P : \mathbb{R}^N\backslash\{0\} \to (0, \infty)$ is called an *m-homogeneous positive function* if $P(x) > 0$ for $x \neq 0$ and $P(tx) = t^m P(x)$ for $t \in [0, \infty)$, $x \in \mathbb{R}^N$.

This is a generalization of a homogeneous polynomial P_{2m} of order $2m$ satisfying $P_{2m}(x) > 0$ for $x \neq 0$. We define a norm $\|u\|_P \equiv \left(\int_{\mathbb{R}^N} |\nabla u|^2 + Pu^2\,dx\right)^{1/2}$, and a space H the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_P$. Then, by (6) and Sobolev inequalities, we see that the space H is continuously imbedded into the space $L^{p+1}(\mathbb{R}^N)$. Moreover, we have the following result.

Proposition 3.2.2. *There exists a minimizer U_P of the following minimization problem:*

$$I_P \equiv \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + Pu^2\,dx \,\Big|\, \int_{\mathbb{R}^N} u^{p+1}\,dx = 1, u \in H \right\}.$$

Proof. Let $\{u_n\}_n$ be a minimizing sequence of I_P. We can assume that *compactness*, *dichotomy* or *vanishing* occurs for $\{(u_n)^{p+1}\}_n$ (see [Lio1, Lio2] and [Str]). From Proposition 3.2.2 and [Lio2, Lemma I.1], we see that *vanishing* does not occur for $\{(u_n)^{p+1}\}$. It is a standard procedure to show that *dichotomy* does not occur. Thus, there exist $\{x_n\}_n \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$ satisfying $\int_{B(x_n,R)} (u_n)^{p+1}\,dx \geqq 1 - \varepsilon$. Note that $\lim_{|x|\to\infty} P(x) = \infty$. Thus, if $\lim_{n\to\infty} |x_n| = \infty$, $\lim_{n\to\infty} \int_{B(x_n,R)} (u_n)^2\,dx = 0$. For some $s \in (0,1)$,

$$\int_{B(x_n,R)} (u_n)^{p+1}\,dx \leqq \left(\int_{B(x_n,R)} (u_n)^2\,dx \right)^s \left(\int_{B(x_n,R)} (u_n)^{2N/(N-2)}\,dx \right)^{1-s}.$$

Then, from the Sobolev imbedding lemma and Proposition 3.2.2, it follows that $\lim_{n\to\infty} \int_{B(x_n,R)} (u_n)^{p+1}\,dx = 0$. This is a contradiction since $\int_{B(x_n,R)} (u_n)^{p+1}\,dx \geqq 1 - \varepsilon$. Thus, we see that $\limsup_{n\to\infty} |x_n| < \infty$. We can assume that w_n converges weakly to some $w_0 \in H$ as $n \to \infty$. Then, it is easy to see that $\|w_0\|_P^2 \leqq \lim_{n\to\infty} \|w_\varepsilon\|_P^2 = I_P$. Moreover, from the boundedness of $\{x_n\}$, we deduce that $\int_{\mathbb{R}^N} w_0^{p+1}\,dx = 1$. Therefore, w_0 is a minimizer of I_P. □

We see that a scaled function $w \equiv (I_P)^{\frac{1}{p-1}} w_0$ satisfies the following equation:

$$\Delta w - Pw + w^p = 0 \qquad \text{in } \mathbb{R}^N. \tag{10}$$

Now we see an asymptotic profile and an energy estimate of v_ε.

Theorem 3.2.3. *Let x_0 be an isolated zero point of V. Suppose that $V(x + x_0) = P(x) + Q(x)$, where P is an m-homogeneous positive function and $\lim_{|x|\to 0} |x|^{-m}|Q(x)| = 0$. Let v_ε be a localized solution of* (3) *around x_0 given in Theorem* 2.1. *Then,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2m}{m+2}\frac{p+1}{p-1} - \frac{2N}{m+2}} \Gamma_\varepsilon(v_\varepsilon) = I_P^{\frac{p+1}{p-1}} (1/2 - 1/(p+1)).$$

Moreover, a rescaled function $\varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} v_\varepsilon(\varepsilon^{\frac{2}{m+2}} x)$ subconverges to a least-energy solution w of (10) *uniformly on $\mathbb{R}^N$.*

Proof. We define

$$w_\varepsilon(x) = \varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} v_\varepsilon(\varepsilon^{\frac{2}{m+2}} x + x_0).$$

Then, we see that

$$\begin{aligned} &\varepsilon^{-\frac{2m}{m+2}\frac{p+1}{p-1} - \frac{2N}{m+2}} \Gamma_\varepsilon(v_\varepsilon) \\ &\quad = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla w_\varepsilon|^2 + \frac{1}{2}(P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x))(w_\varepsilon)^2 - \frac{1}{p+1}(w_\varepsilon)^{p+1}\,dx, \end{aligned} \tag{11}$$

and that

$$\Delta w_\varepsilon(x) - (P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)) w_\varepsilon + w_\varepsilon^p = 0. \tag{12}$$

For a least-energy solution w of (10), it is easy to see that for some $C, c > 0$, $w(x) \leqq C \exp(-c|x|)$, $x \in \mathbb{R}^N$. Then, we see that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} \frac{1}{2}|\nabla w_\varepsilon|^2 + \frac{1}{2}(P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x))(w_\varepsilon)^2 - \frac{1}{p+1}(w_\varepsilon)^{p+1}\,dx \\ \leqq \int_{\mathbb{R}^N} \frac{1}{2}|\nabla w|^2 + \frac{1}{2}P(x)w^2 - \frac{1}{p+1}w^{p+1}\,dx = I_P^{\frac{p+1}{p-1}}\left(\frac{1}{2} - \frac{1}{p+1}\right).\end{aligned} \tag{13}$$

From Theorem 2.1(iii), for each $\delta' > 0$, there exist $C, c > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \geqq \delta'$,

$$w_\varepsilon(x) \leqq C\varepsilon^{-\frac{2}{p-1}\frac{2}{m+2}} \exp(-c\varepsilon^{-\frac{m}{m+2}}|x|). \tag{14}$$

Since $\lim_{|x|\to 0} |x|^{-m}|Q(x)| = 0$, there exists $\delta_0 > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \leqq \delta_0$,

$$|\varepsilon x|^{-\frac{2m}{m+2}} |Q(\varepsilon^{\frac{2}{m+2}} x))| \leqq \tfrac{1}{2}\min\{P(x) \,\big|\, |x| = 1\}.$$

Then, for $|\varepsilon^{\frac{2}{m+2}} x| \leqq \delta_0$,

$$P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x) \geqq \tfrac{1}{2}P(x). \tag{15}$$

Thus, we deduce from elliptic estimates [GT], (13) and (14) that $\|w_\varepsilon\|_{L^\infty}$ is uniformly bounded for small $\varepsilon > 0$. Since $\lim_{|x|\to\infty} P(x) = \infty$, it follows from (13) and (15) that $\lim_{R\to\infty} \int_{R\leqq|x|\leqq\delta_0\varepsilon^{-\frac{2}{m+2}}} (w_\varepsilon)^2\,dx = 0$ uniformly with respect to small $\varepsilon > 0$. Thus, from elliptic estimates [GT] and (14), we see that

$$\lim_{|x|\to\infty} w_\varepsilon(x) = 0 \qquad \text{uniformly with respect to small } \varepsilon > 0.$$

Then, by the comparison principle and (14), we see that for some $C, c > 0$,

$$w_\varepsilon(x) \leqq C\exp(-c|x|) \qquad \text{uniformly with respect to small } \varepsilon > 0. \tag{16}$$

From (12) and (14), we deduce that for some $C, c > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + \left(P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)\right)(w_\varepsilon)^2\,dx \\ &= \int_{\mathbb{R}^N} \frac{1}{p+1}(w_\varepsilon)^{p+1}\,dx \\ &= \int_{|\varepsilon^{\frac{2}{m+2}} x|\leqq\delta_0} (w_\varepsilon)^{p+1}\,dx + \int_{|\varepsilon^{\frac{2}{m+2}} x|\geqq\delta_0} (w_\varepsilon)^{p+1}\,dx \\ &\leqq \int_{|\varepsilon^{\frac{2}{m+2}} x|\leqq\delta_0} (w_\varepsilon)^{p+1}\,dx + C\exp(-c\varepsilon^{-(p+1)}).\end{aligned} \tag{17}$$

In a manner similar to (6), we can deduce from (15) that for some $C > 0$,

$$\int_{|\varepsilon^{\frac{2}{m+2}} x| \leqq \delta_0} (w_\varepsilon)^2 \, dx \leqq C \int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + (P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x))(w_\varepsilon)^2 \, dx. \tag{18}$$

Thus, we see from (17) and (18) that for some $C, c > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + (P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x))(w_\varepsilon)^2 \, dx \\ &\qquad \leqq C \|w_\varepsilon\|_{L^\infty}^{p-1} \int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + (P(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x))(w_\varepsilon)^2 \, dx \\ &\qquad\quad + C \exp(-c\varepsilon^{-(p+1)}). \end{aligned} \tag{19}$$

We claim that $\liminf_{\varepsilon \to 0} \|w_\varepsilon\|_{L^\infty} > 0$.

In fact, if $\liminf_{\varepsilon \to 0} \|w_\varepsilon\|_{L^\infty} = 0$, we deduce from (11) and (19) that for some constants $C, c > 0$,

$$\varepsilon^{-\frac{2m}{m+2}\frac{p+1}{p-1} - \frac{2N}{m+2}} \Gamma_\varepsilon(v_\varepsilon) \leqq C \exp(-c\varepsilon^{-(p+1)}).$$

Then, from the elliptic estimates [GT], we see that $\|v_\varepsilon\|_{L^\infty}$ decays exponentially as $\varepsilon \to 0$. This contradicts Theorem 2.1(ii).

Now, from the elliptic estimates [GT] and (16), we see that w_ε subconverges uniformly to a least-energy solution w of (10). Then, the energy estimate of v_ε comes from (13). This completes the proof. □

3.3. The infinite case

In this subsection, we will investigate the asymptotic behaviour of a localized solution around x_0 when $V(x)$ is typically of the form $\exp(-1/|x - x_0|^m)$. In fact, we shall consider a more general situation where the level sets of V can be non-convex sets, but strictly star-shaped. More precisely, let Ω be a bounded domain in $\mathbb{R}^N$. We assume that there exists a continuous map $r : \mathbb{R}^N \backslash \{0\} \to (0, \infty)$ satisfying

$$\begin{aligned} &x/t \in \mathbb{R}^N \backslash \overline{\Omega}, && t \in (0, r(x)), \\ &x/t \in \partial\Omega, && t = r(x), \\ &x/t \in \Omega, && t \in (r(x), \infty). \end{aligned}$$

It is easy to see that such an Ω should be a strictly star-shaped domain. Then, for any $x \in \mathbb{R}^N \backslash \{0\}$, we can find a unique pair $(r(x), s(x)) \in (0, \infty) \times \partial\Omega$ with $x = r(x)s(x)$.

Definition 3.3.1. A continuous function $b : \mathbb{R}^N \to [0, \infty)$ is called an Ω *quasi-homogeneous function* if

(i) $b(r(x)s(x))$ depends only on $r(x)$;
(ii) $b(r)$ is strictly increasing with respect to $r \in [0, \infty)$; and

(iii)
$$\lim_{r\to 0}\frac{b(cr)}{b(r)}\quad \begin{cases} <1 & \text{for } c<1,\\ >1 & \text{for } c>1.\end{cases}$$

Moreover, a continuous function $a:\mathbb{R}^N\backslash\{0\}\to(0,\infty)$ is called an asymptotically (Ω,b) *quasi-homogeneous function* if there is an Ω quasi-homogeneous function b satisfying $\lim_{|x|\to 0}\frac{a(x)}{b(x)}=1$.

Let $V(x_0)=0$. We can assume that $x_0=0$. We assume that for $|x|\leqq 1$,
$$V(x)=\exp\left(-\frac{1}{a(x)}\right),$$
where a is an asymptotically (Ω,b) quasi-homogeneous function. We define a function
$$g(\varepsilon)=\frac{1}{b^{-1}\left(\frac{-1}{\log\varepsilon^2}\right)},$$
and
$$w_\varepsilon(x)=(\varepsilon g(\varepsilon))^{\frac{-2}{(p-1)}}v_\varepsilon\left(\frac{x}{g(\varepsilon)}\right).$$
Then, it follows that
$$\Delta w_\varepsilon(x)-(\varepsilon g(\varepsilon))^{-2}\,V\!\left(\frac{x}{g(\varepsilon)}\right)w_\varepsilon(x)+w_\varepsilon^p(x)=0;$$
thus, for $|x|\leqq g(\varepsilon)$,
$$\Delta w_\varepsilon(x)-(\varepsilon g(\varepsilon))^{-2}\exp\left(-\frac{1}{a\left(\frac{x}{g(\varepsilon)}\right)}\right)w_\varepsilon(x)+w_\varepsilon^p(x)=0. \tag{20}$$

If b is an Ω quasi-homogeneous function, it is not difficult to show from (ii), (iii) in Definition 3.3.1 above that for some $\alpha>0$, $\lim_{r\to 0}b(r)/r^\alpha=0$. Then, we can also show that
$$\lim_{\varepsilon\to 0}g(\varepsilon)=\infty \qquad\text{and}\qquad \lim_{\varepsilon\to 0}g(\varepsilon)/|\log\varepsilon|^{1/\alpha}=0. \tag{21}$$
From (21), we deduce that for any $C>0$,
$$\lim_{\varepsilon\to 0}\frac{1}{(g(\varepsilon))^2}\exp\left(\frac{C}{b\left(\frac{1}{g(\varepsilon)}\right)}\right)=\lim_{\varepsilon\to 0}\frac{1}{(\varepsilon^C g(\varepsilon))^2}=\infty.$$

Note that
$$\begin{aligned}
&(\varepsilon g(\varepsilon))^{-2}\exp\left(-\frac{1}{a\left(\frac{x}{g(\varepsilon)}\right)}\right)\\
&=\frac{1}{(g(\varepsilon))^2}\exp\left(\log\frac{1}{\varepsilon^2}-\frac{1}{b\left(\frac{r(x)}{g(\varepsilon)}\right)}\frac{b\left(\frac{r(x)}{g(\varepsilon)}\right)}{a\left(\frac{x}{g(\varepsilon)}\right)}\right)\\
&=\frac{1}{(g(\varepsilon))^2}\exp\left(\frac{1}{b\left(\frac{1}{g(\varepsilon)}\right)}\left(1-\frac{b\left(\frac{1}{g(\varepsilon)}\right)}{b\left(\frac{r(x)}{g(\varepsilon)}\right)}\frac{b\left(\frac{r(x)}{g(\varepsilon)}\right)}{a\left(\frac{x}{g(\varepsilon)}\right)}\right)\right).
\end{aligned}$$

Thus, we see from Definition 3.3.1(iii) that for each compact set $B \subset \Omega$,

$$\lim_{\varepsilon \to 0} \max_{x \in B} (\varepsilon g(\varepsilon))^{-2} \exp\left(-\frac{1}{a\left(\frac{x}{g(\varepsilon)}\right)}\right) = 0. \tag{22}$$

Moreover, from Definition 3.3.1(iii), there exists a sufficiently small $D \in (0, 1)$ such that for any $d > 1$,

$$\lim_{\varepsilon \to 0} \min \left\{ (\varepsilon g(\varepsilon))^{-2} \exp\left(-\frac{1}{a\left(\frac{x}{g(\varepsilon)}\right)}\right) \,\middle|\, r(x) \geqq d, |x| \leqq Dg(\varepsilon) \right\} = \infty. \tag{23}$$

We consider the following limiting problem:

$$\begin{aligned} \Delta w + w^p &= 0 \quad \text{in} \quad \Omega, \\ w &> 0 \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{24}$$

From (22), we deduce that

$$\begin{aligned} \limsup_{\varepsilon \to 0} &\left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + (\varepsilon g(\varepsilon))^{-2} V\left(\frac{x}{g(\varepsilon)}\right) (w_\varepsilon)^2 \, dx \right. \\ &\left. - \frac{1}{p+1} \int_{\mathbb{R}^N} (w_\varepsilon)^{p+1} \, dx \right\} \\ &\leqq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1} \, dx \equiv I(\Omega), \end{aligned}$$

where w is a least-energy solution of (24). Then, from the elliptic estimates [GT], we see that if $d > 1$, $\|w_\varepsilon\|_{L^\infty(\{x \in \mathbb{R}^N |\ r(x) \leqq d\})}$ is bounded uniformly for small $\varepsilon > 0$. Moreover, from the elliptic estimates [GT] and (23), we deduce that

$$\lim_{\varepsilon \to 0} \|w_\varepsilon\|_{L^\infty(\{x \in \mathbb{R}^N |\ r(x) > d,\ |x| \leqq Dg(\varepsilon)\})} = 0.$$

From Theorem 2.1(iii), we see that for any $C > 0$,

$$\lim_{\varepsilon \to 0} \sup\{w_\varepsilon(x) \,|\, |x| \geqq Cg(\varepsilon)\} = 0.$$

Thus, $\|w_\varepsilon\|_{L^\infty}$ is uniformly bounded for small $\varepsilon > 0$. Moreover, as in the finite case, we see that $\|w_\varepsilon\|_{L^\infty}$ is uniformly bounded away from 0 for small $\varepsilon > 0$. Therefore, by arguments similar to those used in preceding cases, we obtain the following result.

Theorem 3.3.2. *Let x_0 be an isolated zero point of V. Suppose that $V(x + x_0) = \exp(-\frac{1}{a(x)})$ for $|x| < 1$, where a is an asymptotically (Ω, b) quasi-homogeneous function. Define*

$$g(\varepsilon) = \frac{1}{b^{-1}\left(\frac{-1}{\log \varepsilon^2}\right)}.$$

Let v_ε be a localized solution of (3) *around x_0 given in Theorem* 2.1. *Then,*

$$\lim_{\varepsilon\to 0}(\varepsilon g(\varepsilon))^{-\frac{2(p+1)}{(p-1)}} g(\varepsilon)^{-N}\Gamma_\varepsilon(v_\varepsilon) = I(\Omega).$$

Moreover, for $d > 0$, a rescaled function $(\varepsilon g(\varepsilon)^{-\frac{2}{p-1}} v_\varepsilon(\frac{x}{g(\varepsilon)})$ subconverges to $\overline{w}$ uniformly on $\{x \in \mathbb{R}^N | \operatorname{dist}(x, \partial\Omega) \geqq d\}$ as $\varepsilon \to 0$, where w is a least-energy solution of (24) *and*

$$\overline{w}(x) = \begin{cases} w(x) & \textit{for } x \in \Omega, \\ 0 & \textit{for } x \notin \Omega. \end{cases}$$

Remark 3.3.3. We consider the problem

$$\begin{aligned} \Delta w + w^p &= 0 && \text{in } \Omega_\lambda, \\ w &> 0 && \text{in } \Omega_\lambda, \\ w &= 0 && \text{on } \partial\Omega_\lambda, \end{aligned} \tag{25}$$

where $\Omega_\lambda \equiv B((-1+\lambda, 0, \dots, 0), 1) \cup B((1-\lambda, 0, \dots, 0), 1)$. We can find a least-energy solution w_λ of (25) in the class of even functions. Then, it is easy to see that for sufficiently small $\lambda > 0$, w_λ has exactly two maximum points. Note that for small $\lambda > 0$, Ω_λ is a strictly star-shaped domain. Then, we can easily find an Ω_λ quasi-homogeneous function b which is even, that is, $b(x_1, \dots, x_N) = b(|x_1|, \dots, |x_N|)$. Now, we let $V(x) = \exp(-\frac{1}{b(x)})$ and v_ε be a least-energy solution of (3) in the class of even functions. Then, as in Theorem 3.3.2, we can show that for each $d > 0$, a rescaled function $(\varepsilon g(\varepsilon))^{-\frac{2}{p-1}} v_\varepsilon(\frac{x}{g(\varepsilon)})$ converges to w_λ uniformly on $\{x \in \Omega_\lambda | \operatorname{dist}(x, \partial\Omega) \geqq d\}$ as $\varepsilon \to 0$. Thus, we see that v_ε has at least two local maximum points $h_\varepsilon^1 \neq h_\varepsilon^2$ such that $|h_\varepsilon^1 - h_\varepsilon^2| \leqq \frac{4}{g(\varepsilon)} \to 0$ as $\varepsilon \to 0$. This phenomenon is in high contrast with the case $\inf_{x\in\mathbb{R}^N} V(x) > 0$; in that case, any single-bump solution and any locally least-energy solution concentrating around local minimum points of V has at most one local maximum point (e.g., [W]).

3.4. Asymptotic profiles for least-energy solutions

In this section, we discuss more fine asymptotic profiles for the least-energy solutions given in Theorem 2.2. As we have seen in the preceding subsections, the faster the potential V decays at a zero, the smaller the energy of a localized solution around the zero is. Thus, we can say that as $\varepsilon \to 0$ the least-energy solutions concentrate around the fastest decaying zeros of V. For instance, when $\operatorname{int}(\mathcal{Z})$ is not empty, the least-energy solution v_ε concentrates around a certain connected component A_j of $\operatorname{int}(\mathcal{Z})$, and its normalization $\varepsilon^{-2/(p-1)} v_\varepsilon(x)$ converges to a least-energy solution U_j of problem (7-j) on A_j as in Subsection 3.3.1. Here, the energy of U_j should be a minimum among the energies of least-energy solutions U_i of problem (7-i) on A_i when A_i's are connected components of $\operatorname{int}(\mathcal{Z})$. Moreover, if $\operatorname{int}(\mathcal{Z})$ is empty and each zero of V is of either the finite case or the infinite case, as $\varepsilon \to 0$, the least-energy solutions v_ε concentrate around the fastest decaying zeros of V and its normalizations, depending on the decay order of the zero, converge to a least-energy solution of problems (10) or (24) as in Subsections 3.3.2 and 3.3.3. Furthermore, if there are at least two points at which V has the fastest decay rate, the least-energy solutions concentrate around a point among the points of the fastest decay rate where the smallest energy is attained among the least-energy solutions of normalized limiting problems as in Subsections 3.2 or 3.3. As in the case of localized solutions, it seems that we need more fine analysis to find a finer asymptotic profile of the least-energy solutions for arbitrary types of zeros of V.

4. Some remarks

We have established an asymptotic profile for localized minimal solutions constructed in Theorem 2.1. (We note that in Section 3, the subconvergence for the asymptotic profile can be replaced by the convergence if the normalized limiting problem has a unique positive solution.) The surprising new phenomenon here is that, depending upon the local behaviour of V near a zero set A of V, we obtain a variety of limiting problems which exhibit quite different features. This is in striking contrast with the situation of $\inf V > 0$ under which, as we surveyed in the introduction, there is only one limiting problem. However, the three cases we consider in Section 3 do not cover all possible types of local behaviours of V around a zero set. Thus, it will be very interesting to investigate the asymptotic behaviour of the localized solutions in more complicate situations.

In the flat and the infinite cases, the limiting problems may have many positive solutions depending on the geometry of bounded domain Ω (see [By2-3] and [D]). Thus, we expect that there can be a rich variety of localized solutions concentrating around the zero set of V depending on the geometry of the set.

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References

[ABC] Ambrosetti, A., Badiale, M., Cingolani, S.: Semiclassical states of nonlinear Schrödinger equations. *Arch. Rational Mech. Anal.* **140**, 285–300 (1997)

[By1] Byeon, J.: Existence of many nonequivalent non-radial positive solutions of semilinear elliptic equations on three dimensional annuli. *J. Differential Equations* **136**, 136–165 (1997)

[By2] Byeon, J.: Existence of large positive solutions of some nonlinear elliptic equations on singularly perturbed domains. *Comm. in P. D. E.* **22**, 1731–1769 (1997)

[By3] Byeon, J.: Nonlinear elliptic problems on singularly perturbed domains. *Proc. Roy. Soc. Edinburgh Sect. A* **31**, 1023–1037 (2001)

[By4] Byeon, J.: Standing waves for nonlinear Schrödinger equations with a radial potential. *Nonlinear Analysis, TMA* **50**, 1135–1151 (2002)

[CNY] Cao, D., Noussair, E., Yan, S.: Multiplicity of asymmetric solutions for problems arising in nonlinear optics. Preprint

[D] Dancer, E.N.: The effect of the domain shape on the number of positive solutions of certain nonlinear equations. *J. Differential Equations* **74**, 120–156 (1988)

[DF1] del Pino, M., Felmer, P.L.: Local mountain passes for semilinear elliptic problems in unbounded domains. *Calculus of Variations and PDE* **4**, 121–137 (1996)

[DF2] DEL PINO, M., FELMER, P.L.: Semi-classical states for nonlinear Schrödinger equations. *J. Functional Analysis* **149**, 245–265 (1997)
[DF3] DEL PINO, M., FELMER, P.L.: Multi-peak bound states for nonlinear Schrödinger equations. *Ann. Inst. Henri Poincaré* **15**, 127–149 (1998)
[DY] DANCER, E.N., YAN, S.: On the existence of multipeak solutions for nonlinear field equations on $\mathbf{R}^N$. *Discrete Contin. Dynam. Systems* **6**, 39–50 (2000)
[FW] FLOER, A., WEINSTEIN, A.: Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential *J. Functional Analysis.* **69**, 397–408 (1986)
[G] GUI, C.: Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. *Comm. Partial Differential Equations* **21**, 787–820 (1996)
[GT] GILBARG, D., TRUDINGER, N.S.: *Elliptic Partial Differential Equations of Second Order*. Second edition, Grundlehren 224, Springer (1983)
[KW] KANG, X., WEI, J.: On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Differential Equations* **5**, 899–928 (2000)
[L] LI, Y.Y.: On a singularly perturbed elliptic equation. *Adv. Differential Equations* **2**, 955–980 (1997)
[Lio1] LIONS, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. Part 1. *Ann. Inst. H. Poincaré* **1**, 109–145 (1984)
[Lio2] LIONS, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. Part 2. *Ann. Inst. H. Poincaré* **1**, 223–283 (1984)
[Me] MEYSTRE, P.: *Atom Optics*. Springer (2001)
[Mi] MILLS, D.L.: *Nonlinear optics*. Springer (1998)
[O1] OH, Y.G.: Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$. *Comm. Partial Differential Equations* **13**, 1499–1519 (1988)
[O2] OH, Y.G.: Correction to: Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$. *Comm. Partial Differential Equations* **14**, 833–834 (1989)
[O3] OH, Y.G.: On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* **131**, 223–253 (1990)
[PW] PROTTER, M.H., WEINBERGER, H.F.: *Maximum Principles in Differential Equations*. Springer-Verlag (1984)
[R] RABINOWITZ, P.H.: On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43**, 270–291 (1992)
[Str] STRUWE, M.: *Variational Methods; Application to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer-Verlag (1990)
[W] WANG, X.: On concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.* **153**, 229–244 (1993)
[Wa] WANG, Z.Q.: Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations. *J. Differential Equations* **159**, 102–137 (1999)

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