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Symmetry Breaking of Extremal Functions for the Caffarelli-Kohn-Nirenberg Inequalities

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Abstract

For the L^p $(p > 1)$ version of the Caffarelli-Kohn-Nirenberg inequalities ([4]) we prove a symmetry breaking result for the extremal functions.

1 Introduction

This paper is concerned with the symmetry property of the extremal functions for the following weighted Sobolev inequalities due to Caffarelli, Kohn and Nirenberg in [4] (see also generalization by Lin in [15]). Let $p > 1$ and N the dimension of the space $\mathbf{R}^N$. For all $u \in C_0^\infty(\mathbf{R}^N)$ it holds that

$$\left(\int_{\mathbf{R}^N} |x|^{-bq}|u|^q \, dx\right)^{p/q} \leq C_{a,b} \int_{\mathbf{R}^N} |x|^{-pa}|\nabla u|^p \, dx \tag{1}$$

where, for $N > p$,

$$-\infty < a < \frac{N-p}{p}, \quad 0 \leq b-a \leq 1, \quad \text{and } q = \frac{pN}{N-p+p(b-a)}, \tag{2}$$

and for $N \leq p$,

$$-\infty < a < \frac{N-p}{p}, \quad \frac{p-N}{p} < b-a \leq 1, \quad \text{and } q = \frac{pN}{N-p+p(b-a)}. \tag{3}$$

Let $\mathcal{D}_a^{1,p}(\mathbf{R}^N)$ be the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the following norm

$$||u||_a = \left(\int_{\mathbf{R}^N} |x|^{-pa}|\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{4}$$

Then we see that (1) holds for $u \in \mathcal{D}_a^{1,p}(\mathbf{R}^N)$; this implies that under the conditions (2) and (3), $\mathcal{D}_a^{1,p}(\mathbf{R}^N)$ is embedded continuously into weighted $L^q(\mathbf{R}^N)$ with weight $|x|^{-bq}$. We define

$$S(a,b) = \inf_{u\in\mathcal{D}_a^{1,p}(\mathbf{R}^N)\setminus\{0\}} E_{a,b}(u), \tag{5}$$

to be the best embedding constants, where

$$E_{a,b}(u) = \frac{\displaystyle\int_{\mathbf{R}^N} |x|^{-pa}|\nabla u|^p \, dx}{\left(\displaystyle\int_{\mathbf{R}^N} |x|^{-bq}|u|^q \, dx\right)^{p/q}}. \tag{6}$$

The main result of the paper is

Theorem 1.1 *Let $N > p > 1$ be fixed. Then there is a function $h(a)$ defined for $a \leq 0$ satisfying $h(0) = 0$, $h(a) \in (a, a+1)$ for $a < 0$ and $h(a) - (a+1) \to 0$ as $a \to -\infty$, such that for any (a,b) with $a \leq b < h(a)$ the extremal function for $S(a,b)$, if it exists, is nonradial.*

Theorem 1.2 *Let $p > 1$, $N \leq p$, and $b - a \in (\frac{p-N}{p}, 1)$ be fixed. Then for sufficiently large $-a$, the extremal function for $S(a,b)$ is nonradial.*

Remark 1.3 *The existence of the extremal functions is referred to [7] [11] [13] [16, 17] [20] (see also Remark 3.1).*

Much progress has been made for the case $p = 2$. In [1], [18], the best constants and the minimizers for the Sobolev inequality ($a = b = 0$) were given by Aubin, and Talenti. They proved in this case the minimizers are radial, and found explicitly the minimizers. In [14], Lieb considered the case $a = 0$, $0 < b < 1$ and gave the best constants and explicit radial minimizers. In [9], Chou and Chu considered the a-nonnegative region and gave the best constants and explicit minimizers. Also P. L. Lions [16, 17], and Wang and Willem [19] have established the compactness of all minimizing sequences up to dilations provided $b \geq a \geq 0$. The symmetry of the minimizers has been studied in [9] for $a \geq 0$. In fact, *all* nonnegative solutions in $\mathcal{D}_a^{1,2}(\mathbf{R}^N)$

for the corresponding Euler equation are *radially symmetric* (in the case $a = b = 0$, they are radially symmetric with respect to some point) ([9]) and explicitly given ([1][9][14][18]). The radial symmetry was established in [9], using a generalization of the moving plane method.

Still for $p = 2$, the symmetry breaking phenomena was discovered recently in [7] with $a < 0$, in which a conformal transformation is used to convert the inequality in $\mathbf{R}^N$ to one defined on a cylinder. Then a bifurcation analysis is used to get the symmetry breaking of extremal functions. For $p \neq 2$, the approach in [7] fails. In this paper we shall use a different idea which makes use of the radial extremal functions in the radial class to construct suitable test functions. Then an energy comparison gives the desired result. This approach is in contrast with that used in [3] in which instead of the extremal functions in the radial class they used a certain asymptotic property of the extremal functions in the whole functional space. The method used in this paper is rather robust and should be useful in other symmetry breaking problems (see e.g., [2] and references therein) in which the radial extremal functions develop energy concentrations bounded away from 0.

More precisely, in our setting, first we shall use suitable dilations to make the radial minimizers to develop a sharp layer near $|x| = 1$; so the energy of the dilated minimizers concentrate near $|x| = 1$. Then we take a portion of the radial minimizer as a test function, and the energy can be estimated proportional to the radial energy accordingly, which implies that the best constant $S(a, b)$ is less than the best imbedding constant for the class of radial functions (see $S_R(a, b)$ below), and leads to symmetry breaking.

Theorems 1.1 and 1.2 will be proved in Sections 2 and 3 respectively.

2 Proofs of the main results

The energy functional $E_{a,b}(u)$ is invariant under the following dilations

$$u^\tau(x) = \tau^{\frac{N-p-pa}{p}} u(\tau x), \quad \tau > 0.$$

In fact, both the numerator and the denominator in $E_{a,b}(u)$ are invariant under this dilation. Let us define the best constant in the radial class to be

$$S_R(a, b) = \inf_{u \in \mathcal{D}^{1,p}_{a,R}(\mathbf{R}^N) \setminus \{0\}} E_{a,b}(u),$$

where $\mathcal{D}^{1,p}_{a,R}(\mathbf{R}^N) = \{u \in \mathcal{D}^{1,p}_a(\mathbf{R}^N) \mid u \text{ is radial } \}$. It is well known that for $a < \frac{N-p}{p}$ and $b - a < 1$, $S_R(a,b)$ is always achieved and the extremal functions are given by

$$U_{a,b}(r) = c_0 \left(\frac{N-p-pa}{1 + r^{\frac{p(N-p-pa)(1+a-b)}{(p-1)(N-p(1+a-b))}}} \right)^{\frac{N-p(1+a-b)}{p(1+a-b)}}$$

where

$$c_0 = \left(\frac{N}{(p-1)^{p-1}(N-p(1+a-b))} \right)^{\frac{N-p(1+a-b)}{p^2(1+a-b)}}.$$

We first prove the following result which gives partially Theorem 1.1 and Theorem 1.2.

Theorem 2.1 *Let $p > 1$ and $N \geq 2$ be fixed. While keeping $b-a$ as a constant satisfying* $\max\{0, (p-N)/p\} < b - a < 1$, *for $-a$ sufficiently large, we have $S(a,b) < S_R(a,b)$.*

In order to prove Theorem 2.1, from now on, we fix $N \geq 2, p > 1$ and $b - a \in (\max\{0, (p-N)/p\}, 1)$. We shall consider the asymptotic behaviour of $S(a,b)$ as $-a \to \infty$, comparing with $S_R(a,b)$. Let us call

$$\lambda = \lambda(a) := N - p - pa.$$

Then we see that $\lambda \to \infty$ as $-a \to \infty$. Define

$$d = \frac{N - p(1+a-b)}{p(1+a-b)}.$$

It follows from (2) and (3) that $d > 0$. Then we have an expression

$$U_{a,b}(r) = c_0 \left(\frac{\lambda}{1 + r^{\frac{\lambda}{(p-1)d}}} \right)^d.$$

In order to make $U_{a,b}$ concentrate near $|x| = 1$ we choose a special dilation. For each $\lambda > 0$, we define

$$\tau_\lambda = \lambda^{-\frac{pd}{\lambda}}$$

and

$$v_\lambda(x) = U^{\tau_\lambda}_{a,b}(x) = c_0 \left(\frac{1}{1 + |\tau_\lambda x|^{\frac{\lambda}{(p-1)d}}} \right)^d.$$

It is easy to see that as $\lambda \to \infty$, $\tau_\lambda \to 1$, and that for $|x| < 1$, $v_\lambda(x) \to c_0$ and for $|x| > 1$, $v_\lambda(x) \to 0$.

Lemma 2.2 *For each $r \in (0,1)$, it follows that*

$$\lim_{\lambda\to\infty}\int_{B_{1-r}(0)} |x|^{-ap}|\nabla v_\lambda|^p dx = \lim_{\lambda\to\infty}\int_{\mathbf{R}^N\setminus B_{1+r}(0)} |x|^{-ap}|\nabla v_\lambda|^p dx = 0$$

and

$$\lim_{\lambda\to\infty}\int_{B_{1-r}(0)} |x|^{-bq}|v_\lambda|^q dx = \lim_{\lambda\to\infty}\int_{\mathbf{R}^N\setminus B_{1+r}(0)} |x|^{-bq}|v_\lambda|^q dx = 0.$$

For simplicity of notations, we write $B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$ for $m > 0$ and $n > 0$. From an elementary formula for Gamma function Γ we have

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} > 0.$$

In order to compare $S(a,b)$ with $S_R(a,b)$ we need to calculate $S_R(a,b)$ first.

Lemma 2.3 *Let $d_1 = \frac{N-(1+a-b)}{1+a-b}$. Then*

$$\int_{\mathbf{R}^N} |x|^{-ap}|\nabla v_\lambda(x)|^p dx = \frac{c_0^p d\lambda^{d_1}}{(p-1)^{p-1}} \cdot B((p-1)d+p, d),$$

and

$$\int_{\mathbf{R}^N} |x|^{-bq}|v_\lambda(x)|^q dx = c_0^q(p-1)d\lambda^{d_1} \cdot B\left(\frac{(p-1)N}{p(1+a-b)}, \frac{N}{p(1+a-b)}\right).$$

Finally, we need a couple of technical lemmas. In the following, we use C to denote some constants which do not depend upon λ.

Lemma 2.4 *For $\lambda > p(p-1)$, it holds that*

$$\int_{\mathbf{R}^N} |x|^{-ap}|v_\lambda|^p dx = C\lambda^{d_1-p(1-pd/\lambda)} \cdot B((p-1)d+p(p-1)d/\lambda, d-p(p-1)d/\lambda).$$

Lemma 2.5 *Let $p > 1$. Then there is $C_p > 0$ such that for any $A \geq 0$ and $B \geq 0$*

$$(A+B)^p \leq A^p + B^p + C_p(A^{p-1}B + AB^{p-1}).$$

Lemma 2.5 is a calculus lemma and we omit its proof. The proofs of Lemma 2.2, 2.3 and 2.4 are similar and we give the proof of Lemma 2.4 in next section.

Proof of Theorem 2.1 Choose $\delta \in (0, \frac{1}{2})$ which will be determined later. Let $x_0 = (1, 0, ...0)$ and let η be a cut-off function such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$ and $|\eta'(t)| \leq 2$. Define

$$w_\lambda(x) = \eta(\frac{|x - x_0|}{\delta})v_\lambda(x).$$

We claim that there is a constant C independent of δ satisfying

$$\limsup_{\lambda \to \infty} \frac{E_{a,b}(w_\lambda)}{E_{a,b}(v_\lambda)} \leq C\delta^{\frac{(N-1)(q-p)}{q}}.$$

Then, since $S_R(a, b) = E_{a,b}(v_\lambda)$, for λ large,

$$S(a, b) \leq E_{a,b}(w_\lambda) < E_{a,b}(v_\lambda) = S_R(a, b)$$

if we choose δ small; thus our desired conclusion follows. To prove the claim, we first observe that by Lemma 2.5

$$\begin{aligned}
&\textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla w_\lambda(x)|^p dx \\
&= \textstyle\int_{B_{2\delta}(x_0)} |x|^{-ap}|\eta\nabla v_\lambda(x) + \eta' v_\lambda(x)|^p dx \\
&\leq \textstyle\int_{B_{2\delta}(x_0)} |x|^{-ap}|\nabla v_\lambda(x)|^p dx + \int_{B_{2\delta}(x_0)} |x|^{-ap}|\eta'|^p v_\lambda^p dx \\
&\quad + C_p \textstyle\int_{B_{2\delta}(x_0)} |x|^{-ap}(\eta^{p-1}|\nabla v_\lambda|^{p-1}|\eta'|v_\lambda + \eta|\nabla v_\lambda||\eta'|^{p-1}v_\lambda^{p-1})dx \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Let $A_\delta = \{x \mid 1 - \delta < |x| < 1 + \delta\}$ and $D_\delta = \{(r, \theta) \mid 1 - \delta < r < 1 + \delta,\ |\theta| < \delta\ \}$. Then it is easy to see that there are positive constants C_1 and C_2, independent of $\delta \in (0, \frac{1}{2})$, such that

$$C_1\delta \leq \mathrm{Vol}(A_\delta) \leq C_2\delta$$

and

$$C_1\delta^N \leq \mathrm{Vol}(D_\delta) \leq C_2\delta^N.$$

Here Vol$(\cdot)$ is the Lebesgue measure. From Lemma 2.2 and Lemma 2.3, it follows that

$$\begin{aligned}
I_1 &\leq \tfrac{\mathrm{Vol}(D_{2\delta})}{\mathrm{Vol}(A_{2\delta})} \textstyle\int_{A_{2\delta}} |x|^{-ap}|\nabla v_\lambda(x)|^p dx \\
&\leq C\delta^{N-1} \textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla v_\lambda(x)|^p dx.
\end{aligned} \tag{7}$$

By Lemma 2.4,

$$I_2 \le C(\frac{2}{\delta})^p \lambda^{d_1 - p(1-\frac{pd}{\lambda})}$$

since $|\eta'| \le \frac{2}{\delta}$ and, as $\lambda \to \infty$,

$$B((p-1)d + p(p-1)d\lambda^{-1}, d - p(p-1)d\lambda^{-1}) \to B((p-1)d, d) > 0$$

and $|\eta'| \le \frac{2}{\delta}$. As for I_3, we observe that for any $\epsilon > 0$ there is a positive constant C_ϵ such that

$$\eta^{p-1}|\nabla v_\lambda|^{p-1}|\eta'|v_\lambda \le \epsilon\eta^p|\nabla v_\lambda|^p + C_\epsilon|\eta'|^p v_\lambda^p$$

and

$$\eta|\nabla v_\lambda||\eta'|^{p-1}v_\lambda^{p-1} \le \epsilon\eta^p|\nabla v_\lambda|^p + C_\epsilon|\eta'|^p v_\lambda^p.$$

Thus, from Lemma 2.4, it follows that

$$\begin{aligned} I_3 &\le C_p\epsilon \textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla v_\lambda(x)|^p dx + C_pC_\epsilon(\frac{2}{\delta})^p \int_{\mathbf{R}^N} |x|^{-ap}|v_\lambda(x)|^p dx \\ &= C_p\epsilon \textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla v_\lambda(x)|^p dx + CC_pC_\epsilon(\frac{2}{\delta})^p\lambda^{d_1-p(1-\frac{pd}{\lambda})}. \end{aligned} \tag{8}$$

Thus we see that

$$\begin{aligned} \textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla w_\lambda|^p dx &\le (C\delta^{N-1} + C_p\epsilon + o(1)) \textstyle\int_{\mathbf{R}^N} |x|^{-ap}|\nabla v_\lambda|^p dx \\ &\quad + C_pC_\epsilon(\frac{2}{\delta})^p\lambda^{d_1-p(1-\frac{pd}{\lambda})}. \end{aligned} \tag{9}$$

Here and after, $o(1)$ means that $o(1) \to 0$ as $\lambda \to \infty$. On the other hand,

$$\begin{aligned} \textstyle\int_{\mathbf{R}^N} |x|^{-bq}|w_\lambda|^q dx &= \textstyle\int_{B_{2\delta}(x_0)} |x|^{-bq}\eta^q v_\lambda^q dx \\ &\ge \textstyle\int_{1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \int_{|\theta|<\frac{\delta}{2}} |x|^{-bq} v_\lambda^q dr d\theta \\ &\ge \frac{\mathrm{Vol}(D_{\frac{\delta}{2}})}{\mathrm{Vol}(A_{\frac{\delta}{2}})} \textstyle\int_{A_{\frac{\delta}{2}}} |x|^{-bq} v_\lambda^q dx \\ &\ge C\delta^{N-1}\left(\textstyle\int_{\mathbf{R}^N} |x|^{-bq} v_\lambda^q dx + o(1)\right) \end{aligned} \tag{10}$$

Now, combining estimations (9) and (10), we see that for any $\epsilon > 0$,

$$\limsup_{\lambda\to\infty} \frac{E_{a,b}(w_\lambda)}{E_{a,b}(w_\lambda)} \le C\delta^{(N-1)(1-\frac{p}{q})} + CC_p\epsilon\delta^{-(N-1)p/q}.$$

Since $\epsilon > 0$ is arbitrary, for sufficiently small $\delta > 0$,

$$\limsup_{\lambda\to\infty} \frac{E_{a,b}(w_\lambda)}{E_{a,b}(v_\lambda)} \le C\delta^{(N-1)(1-\frac{p}{q})}.$$

This completes the proof. □

Theorem 2.6 *Let $N > p > 1$ be fixed. Let $a < 0$ be fixed. Then, if $b - a$ is nonnegative and sufficiently small, $S(a,b) < S_R(a,b)$.*

Proof First we observe that both $S(a,b)$ and $S_R(a,b)$ are continuous in (a,b) (this can be proved by similar argument to that used in [7] for the case $p = 2$). Next, fix $z \in \mathbf{R}^N \setminus \{0\}$.. We take any $u \in C_0^\infty(\mathbf{R}^N)$, and compute $E_{a,a}(u^\tau(x-z))$. As $\tau \to 0$, the support of $u^\tau(\cdot - z)$ gets to shrink to the point z. Thus

$$\lim_{\tau\to 0} E_{a,a}(u^\tau(x-z)) = E_{0,0}(|z|^{-a}u(x-z)) = E_{0,0}(u).$$

This shows that $S(a,a) \le S(0,0)$ for all $a < 0$. On the other hand, from Lemma 2.3, we deduce that $S_R(a,a)$ is strictly decreasing for $a \le 0$. Therefore, from the continuity of S and S_R in (a,b) and the fact $S_R(0,0) = S(0,0)$ ([1, 18]), it follows that $S(a,b) < S_R(a,b)$ if $b-a$ is nonnegative and sufficiently small for any fixed $a < 0$. $\Box$

Theorem 2.7 *For $p > N = 1$, $a < (1-p)/p$ and $(p-1)/p < b - a < 1$, it always holds that $S(a,b) < S_R(a,b)$. Moreover, the extremal functions can be taken as 'one half of $U_{a,b}$' in the sense that the extremal functions are equal to $U_{a,b}$ on $(0,\infty)$ (or $(-\infty,0)$) and equal to 0 on $(-\infty,0)$ (or $(0,\infty)$); and $S(a,b) = 2^{-(q-p)/p}S_R(a,b)$.*

Proof. First we can check that $D^{1,p}(\mathbf{R}^1) = \overline{C_0^\infty(\mathbf{R}^1 \setminus \{0\})}$ (as used for $p = 2$ in [7]). For any $u \in D^{1,p}(\mathbf{R}^1)$, we define u_h by $u_h(x) = u(x)$ for $x > 0$ and $u_h(x) = 0$ for $x < 0$. Then,it follows that $u_h \in D^{1,p}(\mathbf{R}^1)$. Therefore,

$$S_R(a,b) = E_{a,b}(U_{a,b}) = 2^{\frac{q-p}{p}} E_{a,b}((U_{a,b})_h);$$

this shows that $S(a,b) < S_R(a,b)$.

To show that the extremal function is $(U_{a,b})_h$, we denote the extremal function by $V_{a,b}$. Then following the same argument as above we have $V_{a,b}$ is zero in $(-\infty,0)$ and $E_{a,b}(V_{a,b}) = E_{a,b}((U_{a,b})_h)$. $\Box$

Finally, we can prove our main results stated in the introduction.

Theorem 1.1 follows from Theorem 2.1 and Theorem 2.6. Theorem 1.2 follows from Theorem 2.1 and Theorem 2.7.

3 The proofs of technical lemmas and further remarks

Recall $v_\lambda(r) = c_0(1 + (\tau_\lambda r)^{\frac{\lambda}{(p-1)d}})^{-d}$. First one has

$$v'_\lambda(r) = -\frac{c_0\lambda}{p-1}\tau_\lambda^{\frac{\lambda}{(p-1)d}}(1 + (\tau_\lambda r)^{\frac{\lambda}{(p-1)d}})^{-d-1} r^{\frac{\lambda}{(p-1)d}-1}.$$

Lemmas 2.2-4 all follow from straight forward computations. We give a proof of Lemma 2.4 here. By a direct computation, for $\lambda > p(p-1)$, we have

$$\begin{aligned}
& \int_{\mathbf{R}^N} |x|^{-ap}|v_\lambda|^p dx \\
= \;& C\int_0^\infty r^{-bp+N-1}(1 + (\tau_\lambda r)^{\frac{\lambda}{(p-1)d}})^{-dp} dr \\
= \;& C\tau_\lambda^{bp-N}\int_0^\infty t^{-bp+N-1}(1 + t^{\frac{\lambda}{(p-1)d}})^{-dp} dt \\
= \;& C\frac{\tau_\lambda^{bp-N}}{\lambda}\int_0^\infty \frac{s^{\frac{(p-1)d(N-bp)}{\lambda}-1}}{(1+s)^{dp}} ds \\
= \;& C\lambda^{\frac{pd(\lambda+p)}{\lambda}-1}\int_0^\infty \frac{s^{\frac{(p-1)d(N-bp)}{\lambda}-1}}{(1+s)^{dp}} ds \\
= \;& C\lambda^{d_1-p(1-\frac{pd}{\lambda})}B(\tfrac{(p-1)d(\lambda+p)}{\lambda}, dp - \tfrac{(p-1)d(\lambda+p)}{\lambda}).
\end{aligned}$$

Remark 3.1 *One open question is whether $S(a,a)$ is achieved for $a < 0$. We have $S(a,a) \le S(0,0)$ for $a < 0$. If $S(a,a) = S(0,0)$ for $a < 0$ it follows that $S(a,a)$ is not achieved for $a < 0$. This is the case when $p = 2$ and it is open for $p \neq 2$.*

Remark 3.2 *For $N > p = 2$ and $a \ge 0$, it is known that all extremal functions of $S(a,b)$ are radially symmetric functions (given by $U_{a,b}$). It would be interesting to know whether such results still hold for $p \neq 2$. More generally, we would also like to know the exact symmetry of the non-radial extremal functions. For $p = 2$, it was proved in [8] that the non-radial extremal functions have $\mathbf{O}(N-1)$ symmetry.*

Remark 3.3 *For $p = 2$, an asymptotic expansion of $S(a,b)$ was given as $\lambda \to \infty$. Here we have a weak version, i.e., from the proof of Theorem 1.1, for $N > p > 1$, it holds*

$$\lim_{\lambda\to\infty}\frac{S(a,b)}{S_R(a,b)} = 0.$$

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