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On the existence of soliton solutions to quasilinear Schrödinger equations

Markus Poppenberg[1] *, Klaus Schmitt[2], and Zhi-Qiang Wang[3] **

[1] Fachbereich Mathematik, Universität Dortmund, D-44221 Dortmund, Germany
email: poppenberg@math.uni-dortmund.de
[2] Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112, USA
email: schmitt@math.utah.edu
[3] Department of Mathematics and Statistics, Utah State University, Logan, UT 84322, USA
email: wang@math.usu.edu

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Summary. Variational techniques are applied to prove the existence of standing wave solutions for quasilinear Schrödinger equations containing strongly singular nonlinearities which include derivatives of the second order. Such equations have been derived as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics. Direct methods of the calculus of variations and minimax methods like the Mountain Pass Theorem are used. The difficulties introduced by the nonconvex functional $\Phi(u) = \int |\nabla u|^2 u^2$ are substantially different from the semilinear case.

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1. Introduction

Quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + V(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z \tag{1}$$

have been derived as models of several physical phenomena. Here $V = V(x)$, $x \in \mathbb{R}^N$, is a given potential, κ is a real constant and f, h are real functions. For instance, in the case $h(s) = s$, we obtain the equation

$$i\partial_t z = -\Delta z + V(x)z - f(|z|^2)z - \kappa(\Delta|z|^2)z \tag{2}$$

which has been called the superfluid film equation in plasma physics by Kurihara in [15] (cf. [16], [17]). In the case $h(s) = (1+s)^{1/2}$, Equation (1) models the

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Correspondence to: M. Poppenberg

self-channeling of a high-power ultra short laser in matter, see [5], [7], [9], [28] and the references in [6]. Equation (1) also appears in plasma physics and fluid mechanics [15], [16], [19], [21], [26], in the theory of Heisenberg ferromagnets and magnons [3], [14], [18], [27], [33], in dissipative quantum mechanics [13] and in condensed matter theory [20]. In the mathematical literature very few results are known about equations of the form (1), (2). Local and global well posedness results concerning the Cauchy problem are given in [6] in the case $h(s) = (1+s)^{1/2}$ and $N = 1, 2, 3$; the case $h(s) = s$ and $N = 1$ is treated in [17], and the case of general functions h is considered in [24] for $N = 1$ and in [25] for arbitrary space dimension $N \geq 1$.

Using variational methods we shall prove the existence of standing wave solutions for quasilinear Schrödinger equations of the form (1), (2). Putting $z(t) = \exp(-i\lambda t)u(x)$ we obtain a corresponding equation of elliptic type which has variational structure. We shall consider elliptic equations of the form

$$- \Delta u + V(x)u - \kappa(\Delta|u|^2)u = \theta|u|^{p-1}u \tag{3}$$

which are derived from (2) with $h(s) = s, f(s) = \theta s^{(p-1)/2},\ p > 1$. Such equations will be shown to have positive solutions for arbitrarily large values of $\theta > 0$. We shall see that the main mathematical difficulties with Equations (1), (2), (3) are caused by the nonlinearity involving second order space derivatives. In the variational formulation these difficulties concern the nonlinear functional

$$\Phi(u) = \int_{\mathbb{R}^N} |\nabla u|^2 u^2 \tag{4}$$

which is homogeneous of order 4 and non-convex. A further problem is caused by the usual lack of compactness since we are dealing with problems defined in all of $\mathbb{R}^N$. The results of this paper show that certain results valid for the semilinear case (see e.g. [4], [10], [22], [23], [30], [34]) are still valid in this more general setting; however, as will be seen, the difficulties introduced by the functional given by (4) are quite different from the semilinear case.

This paper is organized as follows. In Section 2 we introduce the crucial nonlinear functional and provide some properties. In Section 3 direct methods of the calculus of variations are applied to prove existence results in the one dimensional case. Simultaneously, three different cases are considered: The positive potential V may be assumed either to be periodic and bounded or to achieve its upper finite bound at infinity or to tend to infinity when x tends to infinity. The lack of compactness is overcome by applying a lemma due to Brézis-Lieb and a concentration compactness lemma due to P. L. Lions, yielding the existence of positive solutions to Equation (3). In Section 4 the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] is applied to show the existence of solutions of mountain pass type and also to obtain the existence of infinitely many solutions. Finally, in Section 5 the radially symmetric case is studied, providing the existence of nontrivial solutions in higher space dimensions. General results for higher space dimensions, however, are not available at this time.

2. Preliminaries

We consider the quasilinear Schrödinger equation (cf. [15], [16], [17])

$$i\partial_t z = -\Delta z + V_0(x)z - (\Delta|z|^2)z - \theta|z|^{p-1}z \tag{5}$$

where $V_0 = V_0(x)$ is a potential and $\theta > 0,\ p > 1$ are constants. In this paper we treat the case $\kappa > 0$ in Equation (2), and we may assume, without loss of generality, that $\kappa = 1$. We are looking for solutions $z = z(t,x)$ of Equation (5) where $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Writing $z(t) = \exp(-i\lambda t)u(x)$ for $\lambda \in \mathbb{R}$ and putting $V = V_0 - \lambda$, we obtain for the real valued function $u = u(x)$ the equation

$$-\Delta u + V(x)u - (\Delta|u|^2)u = \theta|u|^{p-1}u. \tag{6}$$

We are looking for positive solutions of Equation (6). The weak formulation of (6) is to prove the existence of u in an appropriate function space such that

$$\int_{\mathbb{R}^N} \nabla u\nabla v + \int_{\mathbb{R}^N} V(x)uv + \int_{\mathbb{R}^N} \nabla(u^2)\nabla(uv) = \theta\int_{\mathbb{R}^N} |u|^{p-1}uv \tag{7}$$

for all $v \in C_0^\infty(\mathbb{R}^N)$. (We shall, henceforth write $\int$ for $\int_{\mathbb{R}^N}$, where the dimension N will be understood from the context.) We introduce, formally, the functional

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int V(x)u^2 + \int |\nabla u|^2u^2, \tag{8}$$

whose formal derivative is given by

$$J'(u)v = \int \nabla u\nabla v + \int V(x)uv + \int \nabla(u^2)\nabla(uv). \tag{9}$$

Equation (7) is formally equivalent (cf. [35], 1.12) to

$$J'(u) = \theta I'(u), \quad I(u) = \frac{1}{p+1}\int |u|^{p+1}. \tag{10}$$

Writing

$$\|u\|^2 = \int |\nabla u|^2 + \int V(x)u^2, \quad \Phi(u) = \int |\nabla u|^2u^2 \tag{11}$$

we have

$$J(u) = \frac{1}{2}\|u\|^2 + \Phi(u). \tag{12}$$

In the case $N = 1$ we have the continuous imbedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (and the compact imbedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R})$). Thus Φ is defined on $W^{1,2}(\mathbb{R})$ and

$$|\Phi(u)| \le \|u\|^2_{L^\infty}\|u\|^2_{W^{1,2}} \le C\|u\|^4_{W^{1,2}}. \tag{13}$$

(In the sequel we shall write $\|u\|_q = \|u\|_{L^q}$, for any $1 \le q \le \infty$.)

Lemma 1. *Let $N = 1$ and $V \in L^\infty(\mathbb{R})$. Then I, J are of class C^1 in $W^{1,2}(\mathbb{R})$.*

Proof. This is proved for I in [35], 1.13. It remains to consider Φ. We have

$$|\Phi(u_n) - \Phi(u)| \leq \|u_n\|^2_{W^{1,2}} \|u_n^2 - u^2\|_\infty + \|u_n'^2 - u'^2\|_1 \|u^2\|_\infty. \tag{14}$$

This inequality implies the continuity of Φ, since

$$\|u_n'^2 - u'^2\|_1 \leq \|u_n - u\|_{W^{1,2}} \|u_n + u\|_{W^{1,2}}. \tag{15}$$

Let $\Phi'(u)$ denote the formal derivative of Φ in (9). Then $\Phi'(u)$ defines a linear continuous functional on $W^{1,2}(\mathbb{R})$. For $u, v \in W^{1,2}(\mathbb{R})$ and $h \neq 0$ have

$$\begin{aligned}
&\frac{1}{h}\Big(\Phi(u + hv) - \Phi(u)\Big) - \Phi'(u)v \\
&= h\Big\{ \int u'^2 v^2 + 4u'v'uv + v'^2 u^2 + 2hu'v'v^2 + 2hv'^2 uv + h^2 v'^2 v^2 \Big\}.
\end{aligned}$$

This gives differentiability. To show that $\Phi' : W^{1,2} \to (W^{1,2})'$ (for a Banach space Y we shall always write Y' for its topological dual) is continuous we choose a sequence $u_n \to u$ in $W^{1,2}$ and $v \in W^{1,2}$ with $\|v\|_{W^{1,2}} \leq 1$. We have

$$\begin{aligned}
\Phi'(u_n)v - \Phi'(u)v &= 2\int u_n' u_n \Big((u_n' - u')v + (u_n - u)v'\Big) \\
&+ 2\int \Big(u_n(u_n' - u') + u'(u_n - u)\Big)(u'v + uv').
\end{aligned}$$

The right hand side tends to 0 uniformly for $\|v\|_{W^{1,2}} \leq 1$. This proves the result.

3. Direct Methods

This section will be concerned with the case of space dimension 1.

Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a real potential. We assume that there exists $\delta > 0$ such that

$$V(x) \geq \delta > 0 \quad \text{a.e.} \tag{16}$$

We consider three different cases simultaneously, namely:

$$V \in L^\infty(\mathbb{R}), \quad V \text{ periodic}, \quad p \geq 3. \tag{17}$$

$$V \in L^\infty(\mathbb{R}), \quad V(x) \to \|V\|_{L^\infty}, \text{as } |x| \to \infty, \quad p \geq 3. \tag{18}$$

$$V(x) \to \infty, \text{ as } |x| \to \infty, \quad p > 1. \tag{19}$$

In the first two cases (17), (18) we use the function space $X = W^{1,2}(\mathbb{R})$; in the third case (19) we shall take advantage of the function space

$$X = \Big\{ u \in W^{1,2}(\mathbb{R}) : \int V(x)u^2\,dx < +\infty \Big\}. \tag{20}$$

In all cases the space X is equipped with the norm defined in (11). We consider for $\theta \in \mathbb{R}$ the equation

$$-u'' + V(x)u - (u^2)''u = \theta |u|^{p-1}u, \quad x \in \mathbb{R}. \tag{21}$$

For the corresponding nonlinear functional J defined by (8) (or by (11), (12)) we are looking for a minimizer

$$\mu = \inf\{J(u) : u \in M\} \tag{22}$$

on the constraint set M given by

$$M = \{u \in X : \|u\|_{p+1} = 1\}\,. \tag{23}$$

The following lemma implies, in particular, the weak sequential lower semicontinuity of the functional J. We note that weak convergence (always denoted by $\rightharpoonup$) in X, defined by (20), implies in the third case, (19), weak convergence in $W^{1,2}(\mathbb{R})$, since $X \hookrightarrow W^{1,2}(\mathbb{R})$ and hence $W^{1,2}(\mathbb{R})' \subset X'$.

Lemma 2. *Assume that $u_n \rightharpoonup u$ in $W^{1,2}(\mathbb{R})$. Then*

$$\liminf_n \int (u_n')^2 u_n^2 \geq \int (u')^2 u^2 + \liminf_n \int (u_n' - u')^2 (u_n - u)^2. \tag{24}$$

Proof. We put $v_n = u_n - u$. Since $u_n' = v_n' + u'$ and $u_n = v_n + u$ we obtain

$$\begin{aligned} \int (u_n')^2 u_n^2 \quad &= \quad \int (v_n')^2 v_n^2 + (u')^2 u^2 + (v_n')^2 u^2 + (u')^2 v_n^2 + 2(v_n')^2 v_n u \\ &\qquad +2(u')^2 v_n u + 4v_n' u' v_n u + 2v_n' u' v_n^2 + 2v_n' u' u^2. \end{aligned}$$

The 3rd and 4th summands on the right hand side are nonnegative. Thus it will be enough to show that the 5th through 9th terms all tend to 0, as $n \to \infty$. Let us consider the 5th summand. Since $u(x) \to 0$, as $|x| \to \infty$, for any $u \in W^{1,2}(\mathbb{R})$ there is a number $\epsilon > 0$ such that $|u(x)| \leq \epsilon$ for $|x| \geq R$. We hence have

$$\left| \int (v_n')^2 v_n u \right| \leq \epsilon \int (v_n')^2 |v_n| + \int_{-R}^{R} (v_n')^2 |v_n||u|. \tag{25}$$

Since $v_n \to 0$, uniformly on compact sets in $\mathbb{R}$, and since (v_n) is bounded in $W^{1,2}(\mathbb{R})$, we have proved the assertion for the 5th summand. The same arguments establish the claim for the 6th and 7th term, and also for the 8th term, since for $\epsilon > 0$, there exists $R > 0$ such that $\int_{|x|\geq R} |u'|^2 \leq \epsilon$. Finally the last term

$$2\int v_n' u' u^2 = \frac{2}{3} \int v_n' (u^3)' \to 0, \tag{26}$$

since $v_n \rightharpoonup 0$ in $W^{1,2}(\mathbb{R})$ and $u^3 \in W^{1,2}(\mathbb{R})$. This proves the lemma.

The following Lemma 3 is due to P. L. Lions (cf. [35], Lemma 1.21).

Lemma 3. *Let (u_n) be a bounded sequence in $W^{1,2}(\mathbb{R})$. Assume that there are numbers $q \geq 2$ and $r > 0$ such that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \int_{x-r}^{x+r} |u_n(y)|^q \, dy = 0. \tag{27}$$

Then $u_n \to 0$ in $L^s(\mathbb{R})$, for any $2 < s < \infty$.

Lemma 4. *Assume* (16) *and either* (17) *or* (19). *Then there exists a sequence* $(w_n) \subset M$, *satisfying* $J(w_n) \to \mu$, *such that* $w_n \rightharpoonup w$ *in* $W^{1,2}(\mathbb{R})$, *for some* $w \neq 0$. *In case* (19) *holds, we have* $w \in M$.

Proof. Let $(u_n) \subset M$ be a minimizing sequence for J. Then (u_n) is bounded in X. Since X is reflexive (in the case of (19), observe (16)), we may assume, by passing to a subsequence, that $u_n \rightharpoonup u$ in X. In case (19) holds, the imbedding $X \hookrightarrow L^{p+1}(\mathbb{R})$ is compact, for X defined by (20); this is proved in [2]. This implies that $u_n \to u$ in $L^{p+1}(\mathbb{R})$ and thus $\|u\|_{p+1} = 1$ and $u \in M$. In the first case, (17), we may assume that V is 1–periodic, i.e., $V(x+1) = V(x)$ a.e. in $\mathbb{R}$. We have $\|u_n\|_{p+1} = 1$. Therefore, by Lemma 3, there exist $\epsilon > 0$ and $r \geq 2$ such that

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}} \int_{x-r}^{x+r} |u_n(y)|^{p+1}\, dy \geq \epsilon > 0. \tag{28}$$

We hence may assume that there are integers x_n and $r > 0$ such that

$$\liminf_{n\to\infty} \int_{x_n-r}^{x_n+r} |u_n(y)|^{p+1}\, dy \geq \frac{\epsilon}{2} > 0. \tag{29}$$

Then $w_n(x) = u_n(x + x_n)$ satisfies $J(w_n) = J(u_n)$ and $\|w_n\|_{p+1} = 1$, i.e., the sequence (w_n) is also a minimizing sequence which satisfies

$$\liminf_{n\to\infty} \int_{-r}^{r} |w_n(x)|^{p+1} dx \geq \frac{\epsilon}{2} > 0. \tag{30}$$

Then (w_n) is bounded in $W^{1,2}(\mathbb{R})$. Passing to a subsequence we have $w_n \rightharpoonup w$ in $W^{1,2}(\mathbb{R})$ and thus $w_n \to w$ in $L^{p+1}_{\mathrm{loc}}(\mathbb{R})$. Hence (30) implies $w \neq 0$.

The following Lemma 5 is due to Brézis and Lieb (cf. [35], Lemma 1.32).

Lemma 5. *Let* (u_n) *be a bounded sequence in* $L^q(\mathbb{R}^N)$, *where* $1 \leq q < \infty$, *such that* $u_n \to u$ *almost everywhere in* $\mathbb{R}^N$. *Then*

$$\lim_{n\to\infty} (\|u_n\|_q^q - \|u_n - u\|_q^q) = \|u\|_q^q. \tag{31}$$

Proposition 1. *Assume* (16) *and either* (17) *or* (19). *Then* μ *in* (22) *is achieved by some nonnegative* $u \in M$, *i.e., there is* $0 \neq u \in M$ *with* $J(u) = \mu$ *and* $u \geq 0$.

Proof. This follows in the case of (19) from Lemmas 2 and 4. We assume (17) and choose w_n and w according to Lemma 4. We have to show that $\|w\|_{p+1} = 1$; then $J(w) = \mu$, by Lemma 2. We assume that $0 < \|w\|_{p+1} < 1$. For $v_n = w_n - w$ we may assume that $\|v_n\|_{p+1} < 1$, as follows from Lemma 5. By Lemma 2 we have

$$\begin{aligned}
\mu &\geq J(w) + \liminf_n J(v_n) \\
&= \frac{1}{2}\|w\|^2 + \Phi(w) + \liminf_n (\frac{1}{2}\|v_n\|^2 + \Phi(v_n)) \\
&> \|w\|_{p+1}^4 J(w/\|w\|_{p+1}) + \liminf_n \|v_n\|_{p+1}^4 J(v_n/\|v_n\|_{p+1}) \\
&\geq \mu \left((\|w\|_{p+1}^{p+1})^{4/(p+1)} + \liminf_n (\|v_n\|_{p+1}^{p+1})^{4/(p+1)} \right) \\
&\geq \mu,
\end{aligned}$$

which is a contradiction. Here the last inequality follows, since $p \geq 3$ and

$$\|w\|_{p+1}^{p+1} + \lim_n \|v_n\|_{p+1}^{p+1} = 1, \tag{32}$$

by Lemma 5. We have proved that $\|w\|_{p+1} = 1$ and thus $w \in M$. Finally, we observe that for a minimizing sequence (w_n) the sequence $(|w_n|)$ is a minimizing sequence, as well (cf. [12], Lemma 7.6). This gives the result.

Theorem 1. *Assume* (16) *and either* (17) *or* (18) *or* (19). *Then there exists* $\theta > 0$ *such that Equation* (21) *admits a positive solution* $u \in W^{1,2}(\mathbb{R})$. *This solution is a ground state solution, i.e., it has lowest energy.*

Proof. We first consider the cases (17), (19). By Proposition 1, there exists $u \in W^{1,2}(\mathbb{R})$ with $u \geq 0$ and $\|u\|_{p+1} = 1$ and a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$J'(u)v = -\int u''v + \int V(x)uv - \int (u^2)''uv = \theta \int |u|^{p-1}uv, \tag{33}$$

for all $v \in W^{1,2}(\mathbb{R})$. In particular, u is a distributional solution of Equation (21). Putting $v = u$ in (33) we see that

$$0 < 2\mu < \theta = J'(u)u = J(\sqrt{2}u) < 4\mu. \tag{34}$$

On the other hand, Equation (21) can be written in the form

$$-(1+2u^2)u'' = \Big(\theta|u|^{p-1} + 2(u')^2 - V(x)\Big)u, \tag{35}$$

which implies that $u'' \in L^1_{\mathrm{loc}}(\mathbb{R})$. Hence u' is (locally absolutely) continuous and so u is of class C^1. To show positivity we assume that $u(x_0) = 0$ for some x_0. Then $u'(x_0) = 0$ since $u \geq 0$. We see that $-u'' = P(x)u$ near x_0 with some L^1_{loc}–function P. Hence $u = 0$ near x_0, and u will vanish identically, which is impossible, since $u \in M$. This establishes the cases (17), (19). In the case (18) let $\bar{V}$ denote the constant potential $\bar{V} = \|V\|_{L^\infty}$ and define $\bar{J}$ and $\bar{\mu}$ by (8), (22) using $\bar{V}$ in place of V in (8) and using $\bar{J}$ in place of J in (22). Applying the above case (17) to the constant potential $\bar{V}$ we can choose a positive function $\bar{u} \in M$ satisfying $\bar{J}(\bar{u}) = \bar{\mu}$. If $V \not\equiv \bar{V}$ we see that $\mu < \bar{\mu}$ since $V \leq \bar{V}$ a.e. and $\bar{u}$ is positive. Let (u_n) be a sequence in M such that $J(u_n) \to \mu$ where we may assume that $u_n \rightharpoonup u_0$. If $u_0 = 0$ then we get from (18) that

$$\mu = \lim_n J(u_n) = \lim_n \bar{J}(u_n) \geq \bar{\mu} \tag{36}$$

which is a contradiction. Thus $u_0 \neq 0$. The proof of Proposition 1 shows that $\|u_0\|_{p+1} = 1$. Hence $u_0 \in M$ and $J(u_0) = \mu$. As above, this gives the assertion.

Next we improve the existence statement on θ. The following theorem shows that there exist solutions for arbitrarily large values of θ.

Theorem 2. *Assume* (16) *and either* (17) *or* (18) *or* (19). *Then there is a sequence* $\theta_n \to +\infty$ *such that Equation* (21) *admits for any* $\theta = \theta_n$ *a positive solution* $u \in W^{1,2}(\mathbb{R})$.

Proof. Let $\alpha > 0$. As in Proposition 1 and Theorem 1 we can choose a positive solution u_α with $\|u\|_{p+1} = \alpha$ of Equation (21) where $\theta = \theta_\alpha$ for some $\theta_\alpha > 0$. It is enough to show that (θ_α) is unbounded. Let us assume that θ_α is bounded as $\alpha \to 0$. Then $u_\alpha \to 0$ in X as $\alpha \to 0$ since $\|u_\alpha\|^2 \le \theta_\alpha \alpha^{p+1}$. However, we have

$$\|u_\alpha\|^2 \le \theta_\alpha \|u_\alpha\|_{p+1}^{p+1} \le C\theta_\alpha \|u_\alpha\|^{p+1} \tag{37}$$

by the continuous imbedding $X \hookrightarrow L^{p+1}$, which is a contradiction, since $p > 1$. This proves the result.

In order to get more precise results, we consider, for $\alpha > 0$, the functional

$$J_\alpha(u) = J(\alpha u) = \frac{\alpha^2}{2}\|u\|^2 + \alpha^4 \Phi(u) \tag{38}$$

and put

$$\mu_\alpha = \inf\{J(u) : u \in \alpha M\} = \inf\{J_\alpha(u) : u \in M\}. \tag{39}$$

It is easy to see that μ_α is strictly increasing and continuous for $\alpha > 0$. We can choose $u_\alpha \in \alpha M$ and $\theta = \theta(\alpha) > 0$ such that $J(u_\alpha) = \mu_\alpha$ and u_α is a positive solution of Equation (21). This implies

$$\theta \alpha^{p+1} = J'(u_\alpha)u_\alpha = J(\sqrt{2}u_\alpha). \tag{40}$$

Note that θ in (40) does not define a function of α since θ in (40) depends on the choice of u_α. However, the minimal θ in (40) defines a function of α by

$$\theta_\alpha = \alpha^{-p-1} \inf\{J_\alpha(\sqrt{2}u) : u \in M, J_\alpha(u) = \mu_\alpha\}. \tag{41}$$

We show that the infimum in (41) is achieved.

Proposition 2. *Assume* (16) *and either* (17) *or* (18) *or* (19). *Let* $\alpha > 0$. *Then there exists* $u \in M$ *such that*

$$J_\alpha(u) = \mu_\alpha, \quad \theta_\alpha = \alpha^{-p-1} J_\alpha(\sqrt{2}u). \tag{42}$$

The function $\alpha \mapsto \theta_\alpha,\ \alpha > 0,$ *is lower semicontinuous.*

Proof. We choose $u_n \in M$ such that $J_\alpha(u_n) = \mu_\alpha$ and $J_\alpha(\sqrt{2}u_n) \to \alpha^{p+1}\theta_\alpha$. Then (u_n) is bounded in X and we may assume by passing to a subsequence that $u_n \rightharpoonup u_0$ in X. As in the proofs of Lemma 4 and Theorem 1 we may assume that $u_0 \neq 0$. Let $v_n = u_n - u_0$. ¿From Lemma 2 we obtain

$$\begin{aligned}\mu_\alpha &\ge \lim_n \|u_0\|_{p+1}^4 J_\alpha(u_0/\|u_0\|_{p+1}) + \lim_n \|v_n\|_{p+1}^4 J_\alpha(v_n/\|v_n\|_{p+1}) \\ &\ge \mu_\alpha(\|u_0\|_{p+1}^4 + \lim_n \|v_n\|_{p+1}^4) \ge \mu_\alpha.\end{aligned}$$

Thus $\mu_\alpha \neq 0$, since $u_0 \neq 0$. In the case of (19) we have $\|u_0\|_{p+1} = 1$ and $\|v_n\|_{p+1} \to 0$; in cases (17), (18) this holds as well by means of the previous estimates. Since

$$\mu_\alpha \ge J_\alpha(u_0) + \lim_n J_\alpha(v_n) \ge \mu_\alpha + \lim_n J_\alpha(v_n), \tag{43}$$

we conclude that $J_\alpha(v_n) \to 0$ and $\|v_n\| \to 0$. Hence $u_0 \in M$ and $J_\alpha(u_0) = \mu_\alpha$. Since J is continuous we obtain $\alpha^{p+1}\theta_\alpha = \lim_n J(\sqrt{2}u_n) = J(\sqrt{2}u_0)$ proving

(42). To show that θ_α is lower semicontinuous let $\alpha_n \to \alpha$ and assume that $\theta_{\alpha_n} \to \theta$ for some θ. Note that $\mu_{\alpha_n} \to \mu_\alpha$. We have to show that $\theta \geq \theta_\alpha$. We choose $u_n \in M$ such that $J_{\alpha_n}(u_n) = \mu_{\alpha_n}$ and $\theta_{\alpha_n}\alpha_n^{p+1} = J_{\alpha_n}(\sqrt{2}u_n)$. As above, we may assume that $u_n \rightharpoonup u_0$ for some $u_0 \neq 0$, where $\|u_0\|_{p+1} = 1$ and $\|v_n\| \to 0$. Since J is continuous we get $J_\alpha(u_0) = \lim_n J_{\alpha_n}(u_n) = \mu_\alpha$ and $\theta_{\alpha_n} = \alpha_n^{-p-1}J_{\alpha_n}(\sqrt{2}u_n) \to \alpha^{-p-1}J_\alpha(\sqrt{2}u_0)$. Hence $\theta = \alpha^{-p-1}J_\alpha(\sqrt{2}u_0)$ and thus $\theta \geq \theta_\alpha$.

We next obtain that Equation (21) admits a positive solution for any parameter $\theta = \theta_\alpha$ and prove some properties of the function $\alpha \mapsto \theta_\alpha$. It should be emphasized that the exact value of θ_α is given by (41).

Theorem 3. *Assume* (16) *and either* (17) *or* (18) *or* (19). *Then for any* $\alpha > 0$ *there is a positive solution* $u \in W^{1,2}(\mathbb{R})$ *of Equation* (21) *with* $\theta = \theta_\alpha$. *The function* $\alpha \mapsto \theta_\alpha$ *is unbounded and lower semicontinuous. In case* $p \geq 3$ *we have*

$$\lim_{\alpha\to+\infty} \theta_\alpha = 0, \qquad \lim_{\alpha\to 0^+} \theta_\alpha = +\infty. \tag{44}$$

Proof. We see that $\mu_\alpha \leq \alpha^4\mu_1$, if $\alpha \geq 1$ and that $\mu_\alpha \geq \alpha^4\mu_1$, if $\alpha \leq 1$. We have

$$2\mu_\alpha < J'(u_\alpha)u_\alpha = J(\sqrt{2}u_\alpha) < 4\mu_\alpha, \tag{45}$$

where $u_\alpha \in \alpha M$ satisfies $J(u_\alpha) = \mu_\alpha$ and $J(\sqrt{2}u_\alpha) = \theta_\alpha\alpha^{p+1}$. Thus we get

$$\theta_\alpha \leq \frac{4\mu_\alpha}{\alpha^{p+1}} \leq \frac{4\alpha^4\mu_1}{\alpha^{p+1}} = \frac{4\mu_1}{\alpha^{p-3}}, \quad \alpha \geq 1, \tag{46}$$

$$\theta_\alpha \geq \frac{2\mu_\alpha}{\alpha^{p+1}} \geq \frac{2\alpha^4\mu_1}{\alpha^{p+1}} = \frac{2\mu_1}{\alpha^{p-3}}, \quad \alpha \leq 1. \tag{47}$$

This implies (44), if $p > 3$. Let $p = 3$ and fix $\theta > 0$. We choose $u_0 \in X$ satisfying $\Phi(u_0) < \theta\|u_0\|_4^4$; e.g., we may let $u_0(x) = \phi_R(x)u_1(\alpha x)$, where $1/R = \alpha > 0$ is sufficiently small, u_1 is a smooth positive function and $\phi_R \in C_0^\infty(\mathbb{R})$ is a cutoff function satisfying $\phi_R(x) = 1$, if $|x| \leq R$, and $\phi_R(x) = 0$, if $|x| \geq R+1$. We may assume that $\|u_0\|_4 = 1$. There exists α_0 such that for any $\alpha \geq \alpha_0$ we have

$$\mu_\alpha \leq J_\alpha(u_0) = \frac{1}{2}\alpha^2\|u\|_0^2 + \alpha^4(\Phi(u_0) - \theta) + \theta\alpha^4 \leq \theta\alpha^4. \tag{48}$$

Hence (46) implies that $\theta_\alpha \to 0$ as $\alpha \to \infty$. Now Theorem 2 gives the result.

Remark 1. In the proof of Lemma 4 we have used the compact imbedding $X \hookrightarrow L^{p+1}(\mathbb{R})$ provided that $V(x) \to \infty$ as $|x| \to \infty$. More generally, the compactness of the imbedding $X \hookrightarrow L^{p+1}(\mathbb{R})$ is proved in [2] under the weaker assumption that the set $\{x : V(x) \leq b\}$ has a finite Lebesgue measure for any $b > 0$; here X is defined by (20).

4. Mountain Pass Methods

Throughout this section we assume that $N = 1$ and fix $p \geq 3$ and $\theta > 0$. Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy (16), (19). Let X, defined by (20), be equipped with the norm (11). We now denote by J^θ the nonlinear functional

$$J^\theta(u) = \frac{1}{2}\int (u')^2 + \frac{1}{2}\int V(x)u^2 + \int (u')^2u^2 - \frac{\theta}{p+1}\int (u_+)^{p+1}, \tag{49}$$

where $u_+ = \max\{u, 0\}$. Note that $J^\theta(u) = \frac{1}{2}\|u\|^2 + \Phi(u) - I^\theta(u)$ where we put $I^\theta(u) = \frac{\theta}{p+1}\int (u_+)^{p+1}$. We have

$$(J^\theta)'(u)v = \int u'v' + \int V(x)uv + \int (u^2)'(uv)' - \theta \int (u_+)^p v \tag{50}$$

(cf. [35], 1.12). We shall use the following version of the mountain pass theorem of [1] as given in [35], Theorems 2.9, 2.10.

Lemma 6. *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ with $J(0) = 0$. Suppose that the following conditions hold:*

(i) *There exist $\rho > 0$ and $r > 0$ such that $J(u) \geq r$ for all $u \in X$ with $\|u\| = \rho$.*
(ii) *There exists $e \in X$ with $\|e\| > \rho$ such that $J(e) \leq 0$.*

Let

$$\Gamma = \Big\{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\Big\} \tag{51}$$

and put

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)). \tag{52}$$

Then $c \geq r$ and there is a sequence (u_n) in X such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0. \tag{53}$$

Lemma 7. *The functional J^θ satisfies the assumptions of* Lemma 6.

Proof. There exists $C > 0$ such that $|\Phi(u) - I^\theta(u)| \leq C\|u\|^4$ for all $u \in X$ with $\|u\| \leq 1$. For $\|u\| = \rho$ we obtain

$$J^\theta(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^4 = \frac{1}{2}\rho^2 - C\rho^4 > 0 \tag{54}$$

if ρ is sufficiently small. On the other hand, let us choose $u_0 \in X$ with $u_0 \neq 0$ arbitrarily in the case $p > 3$, and satisfying $\Phi(u_0) < I^\theta(u_0)$ in the case $p = 3$ (cf. the proof of Theorem 3). We see that

$$J^\theta(\mu u_0) = \frac{1}{2}\mu^2\|u_0\|^2 + \mu^4\Phi(u_0) - \mu^{p+1}I^\theta(u_0) \leq 0, \tag{55}$$

if $\mu > 0$ is large enough. This proves the lemma.

Lemma 8. *Any sequence $(u_n) \subset X$ satisfying* (53) *is bounded in X.*

Proof. We have

$$o(\|u_n\|) = (J^\theta)'(u_n)u_n = \|u_n\|^2 + 4\Phi(u_n) - (p+1)I^\theta(u_n), \tag{56}$$

$$c + o(1) = J^\theta(u_n) = \frac{1}{2}\|u_n\|^2 + \Phi(u_n) - I^\theta(u_n). \tag{57}$$

This implies the assertion, since

$$(p+1)c + o(1) + o(\|u_n\|) = \frac{p-1}{2}\|u_n\|^2 + (p+1-4)\Phi(u_n). \tag{58}$$

Lemma 9. *Let* $(u_n) \subset X$ *satisfy* (53) *for* $c > 0$ *and* $u_n \rightharpoonup u$ *in* $W^{1,2}(\mathbb{R})$. *Then* $u_n \to u$ *in* X, *and* $u(\neq 0)$ *is a critical point of* J^θ, *i.e.,* $(J^\theta)'(u) = 0$.

Proof. Since the imbedding $X \hookrightarrow L^{p+1}(\mathbb{R})$ is compact we have $u_n \to u$ in $L^{p+1}(\mathbb{R})$. Therefore, Hölder's inequality implies that

$$\int |u_n|^p |u_n - u| \to 0, \text{ as } n \to \infty. \tag{59}$$

By assumption we have $(J^\theta)'(u_n)(u_n - u) = o(\|u_n - u\|)$. ¿From (59) we get

$$(J^\theta)'(u_n)(u_n - u) = \|u_n - u\|^2 + 2\int (u_n')^2 (u_n - u)^2 + 2\int u_n^2 (u_n' - u')^2 + o(1). \tag{60}$$

We hence obtain $\|u_n - u\| \to 0$, as $n \to \infty$.

Actually, we have proved in Lemmas 8, 9 that the functional J^θ satisfies the Palais-Smale condition $(\mathrm{PS})_c$ (cf. [35], 1.16) for any $c > 0$.

Theorem 4. *Let* $N = 1$, $p \geq 3$ *and* $\theta > 0$. *Assume* (16), (19). *Then Equation* (21) *admits a positive solution* $u \in X$. *Moreover, the equation admits infinitely many solutions* u_n *satisfying* $J_\theta(u_n) \to \infty$ *as* $n \to \infty$.

Proof. By Lemmas 6 and 7 there exists a sequence $(u_n) \subset X$ satisfying (53), for some $c > 0$. By Lemma 8 we may assume that $u_n \rightharpoonup u$ in $W^{1,2}(\mathbb{R})$. Hence Lemma 9 gives a nontrivial solution which is C^1 by Theorem 1. If we assume that u is negative somewhere then there exists a local minimum x_0 such that $u(x_0) < 0 = u'(x_0) \leq u''(x_0)$ which contradicts Equation (35). This gives a positive solution (cf. Theorem 1). Applying the Fountain Theorem 3.6 of [35], we obtain, in analogy to the semilinear case [35], the existence of infinitely many solutions.

5. The Radially Symmetric Case

In this section we consider the case $N \geq 2$. A function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is called radial if $Tu = u$ for any orthogonal transformation T in $\mathbb{R}^N$ where $Tu(x) = u(T^{-1}x)$. We assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a radial potential satisfying (16) for some $\delta > 0$. (We shall see that we do not need to assume any conditions on V concerning periodicity or unboundedness near infinity.) Let

$$H^1_r(\mathbb{R}^N) = \left\{ u \in W^{1,2}(\mathbb{R}^N) : u \text{ is radial} \right\} \tag{61}$$

denote the subspace of $W^{1,2}(\mathbb{R}^N)$ consisting of all radially symmetric functions. The following Lemma was first proved by Strauss [31] (cf. [35], 1.26).

Lemma 10. *Let $N \geq 2$ and $2 < q < \frac{2N}{N-2}$. Then the imbedding*

$$H_r^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \tag{62}$$

is compact.

The following two lemmas are easily established.

Lemma 11. *For any $N \geq 2$, there is a constant $C = C_N > 0$ such that*

$$|u(x)| \leq C|x|^{(1-N)/2} \|u\|_{W^{1,2}}, \tag{63}$$

for any $|x| \geq 1$ and $u \in H_r^1(\mathbb{R}^N)$.

Lemma 12. *Let $(u_n) \subset H_r^1(\mathbb{R}^N)$ satisfy $u_n \rightharpoonup u_0$ in $W^{1,2}(\mathbb{R}^N)$. Let $R > 0$ and $\epsilon > 0$ be fixed. We then have*

$$\max_{R \geq |x| \geq \epsilon} |u_n(x) - u_0(x)| \to 0, \text{ as } n \to \infty. \tag{64}$$

Lemma 13. *Let $(u_n) \subset H_r^1(\mathbb{R}^N)$ satisfy $u_n \rightharpoonup u_0$ in $W^{1,2}(\mathbb{R}^N)$. Then*

$$\liminf_n \int |\nabla u_n|^2 u_n^2 \geq \int |\nabla u_0|^2 u_0^2. \tag{65}$$

Proof. We write $v_n = u_n - u_0$ and fix $\epsilon > 0$. We have

$$\int |\nabla u_n|^2 u_n^2 \geq \int_{|x| \geq \epsilon} |\nabla(u_0 + v_n)|^2 (u_0 + v_n)^2. \tag{66}$$

We expand the right hand side into nine terms. Besides $\int_{|x|\geq\epsilon} |\nabla u_0|^2 u_0^2$, every term is either nonnegative or tends to 0 as $n \to \infty$. Here we have used (63), (64) and have observed that (u_n) is bounded in $W^{1,2}(\mathbb{R}^N)$ and that

$$\int_{|x| \geq \epsilon} \nabla v_n \nabla u_0 u_0^2 = \frac{1}{3} \int_{|x| \geq \epsilon} \nabla v_n \nabla u_0^3 \to 0 \quad \text{as} \quad n \to \infty, \tag{67}$$

since $u_n \rightharpoonup u_0$ and $u_0^3 \in W^{1,2}(\mathbb{R}^N \setminus B_\epsilon(0))$. Inequality (66) then gives the result.

We define J by (4), (11), (12). Let $1 < p < \frac{N+2}{N-2}$. Put

$$M_1 = \left\{ u \in H_r^1(\mathbb{R}^N) : \|u\|_{p+1} = 1 \right\}, \tag{68}$$

and let

$$\mu_1 = \inf\{J(u) : u \in M_1\}. \tag{69}$$

Note that we do not claim that J is well-defined nor of class C^1 in $H_r^1(\mathbb{R}^N)$.

Proposition 3. *There exists a minimizer $u_0 \in M_1$ such that $J(u_0) = \mu_1$.*

Proof. Let (u_n) be a minimizing sequence for J on M_1. Then (u_n) is bounded in $W^{1,2}(\mathbb{R}^N)$ and we may assume that $u_n \rightharpoonup u_0$ in $W^{1,2}(\mathbb{R}^N)$. It follows from Lemma 10 that $u_n \to u_0$ in $L^{p+1}(\mathbb{R}^N)$. In particular, we have $\|u_0\|_{p+1} = 1$ and $u_0 \in M_1$. Since $u_n \rightharpoonup u_0$ in $W^{1,2}(\mathbb{R}^N)$, we obtain from Lemma 13 the estimates

$$\begin{aligned}
\mu_1 &= \lim_n \left\{ \frac{1}{2}\|u_n\|^2 + \int |\nabla u_n|^2 u_n^2 \right\} \\
&\geq \lim_n \left\{ \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u_n - u_0\|^2 + \int |\nabla u_0|^2 u_0^2 \right\} \\
&\geq J(u_0) + \frac{1}{2} \lim_n \|u_n - u_0\|^2 \\
&\geq J(u_0) \\
&\geq \mu_1
\end{aligned}$$

and hence $\mu_1 = J(u_0)$ and $\lim\limits_n \|u_n - u_0\| = 0$.

Theorem 5. *Let $N \geq 2$ and $1 < p < \frac{N+2}{N-2}$. Then there exist $\theta > 0$ and a nonnegative nontrivial radially symmetric solution $u \in H_r^1(\mathbb{R}^N)$ of Equation* (3) *which is C^1 and positive in $\mathbb{R}^N \setminus \{0\}$.*

Proof. We can choose a nonnegative radially symmetric function $u \in H_r^1(\mathbb{R}^N)$ satisfying $J(u) = \mu_1$ and $u \in M_1$. We note that $\int |\nabla u|^2 u^2$ is finite. Hence the integral $\int \nabla(u^2)\nabla(uv)$ exists for any $v \in C_0^\infty(\mathbb{R}^N)$ because

$$\int |u||\nabla u|^2 + u^2|\nabla u| \leq \int (1+u^2)|\nabla u|^2 + u^2(1+|\nabla u|^2) < +\infty. \tag{70}$$

Therefore, the standard proof of Ljusternik's Theorem on Lagrange multipliers (cf. [11], 1.8.1 or [8]) yields the existence of a real number θ such that

$$\int \nabla u \nabla v + \int V(x)uv + \int \nabla(u^2)\nabla(uv) = \theta \int |u|^{p-1}uv, \tag{71}$$

for all $v \in C_0^\infty(\mathbb{R}^N)$ and for $v = u$ as well. We have $0 < 2\mu_1 < \theta = J(\sqrt{2}u) < 4\mu_1$ and u is a distributional solution of Equation (3). Since u solves an ordinary differential equation where 0 is the only singularity we obtain, as in the proof of Theorem 1, that u is C^1 and thus positive in $\mathbb{R}^N \setminus \{0\}$.

Employing the arguments used in the proofs of Theorems 2, 3 we obtain:

Theorem 6. *Let $N \geq 2$ and $1 < p < \frac{N+2}{N-2}$. Then there exists a sequence (θ_n), $\theta_n \to \infty$, such that Equation* (3) *admits for any $\theta = \theta_n$ a nontrivial nonnegative radial solution $u \in H_r^1(\mathbb{R}^N)$. In case $3 \leq p < \frac{N+2}{N-2}$, i.e. $N = 2$ or $N = 3$, there exists a corresponding sequence (θ_n), satisfying $\theta_n \to 0$, as well.*

In case $N = 3$ we get solutions for any $\theta > 0$ using the mountain pass theorem.

Theorem 7. *Let $N = 3$ and $3 \leq p < 5$ and $\theta > 0$. Then Equation* (3) *admits a nontrivial nonnegative radial solution $u \in H_r^1(\mathbb{R}^3)$.*

Proof. For $\nu > 0$ we define the nonlinear functional

$$I_\nu(u) = \frac{1}{2}\|u\|^2 + \int_{|x|\geq\nu} |\nabla u|^2 u^2 - \frac{\theta}{p+1}\int |u|^{p+1}. \tag{72}$$

By Lemmas 11, 12 we can see that $I_\nu \in C^1(H_r^1(\mathbb{R}^3), \mathbb{R})$. The proof of Lemma 7 gives assumptions (i), (ii) of Lemma 6 for I_ν. The Palais–Smale condition follows for I_ν from the proofs of Lemmas 8, 9. The mountain pass theorem yields a nontrivial critical point $u_\nu \in H_r^1(\mathbb{R}^3)$ for I_ν. The critical values $b_\nu = I_\nu(u_\nu)$ satisfy $0 < b_\mu \leq b_\nu \leq M$ for all $0 < \nu < \mu$ with a constant $M > 0$. We thus have

$$\|u_\nu\|^2 + \int_{|x|\geq\nu} |\nabla u_\nu|^2 u_\nu^2 + (\|u_\nu\|_{L^{p+1}})^{-1} \leq C \tag{73}$$

for all $0 < \nu \leq 1$ and some constant $C > 0$. Extracting a subsequence, we may assume that $u_\nu \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^3)$ and $u_\nu \to u_0$ strongly in $L^{p+1}(\mathbb{R}^3)$ as $\nu \to 0$ where $u_0 \neq 0$ by (73). As in Lemma 13 we also see that $\int |\nabla u_0|^2 u_0^2$ is finite. It remains to show that u_0 satisfies Equation (3) in the distributional sense, that is,

$$\int \nabla u_0 \nabla v + \int V u_0 v + 2\int u_0 \nabla u_0 \nabla(u_0 v) = \theta \int |u_0|^{p-1} u_0 v, \tag{74}$$

for all $v \in C_0^\infty(\mathbb{R}^3)$. By construction of u_0 this holds for all $v \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. We fix $v \in C_0^\infty(\mathbb{R}^3)$ and choose $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(x) = 1$ if $|x| \leq 1/2$ and $\phi(x) = 0$ if $|x| \geq 1$ and put $\phi_\epsilon(x) = \phi(x/\epsilon)$. Since (74) holds for $(1-\phi_\epsilon)v$, it suffices to show that

$$\int \nabla u_0 \nabla(\phi_\epsilon v) + \int V u_0 \phi_\epsilon v + 2\int u_0 \nabla u_0 \nabla(u_0 \phi_\epsilon v) - \theta \int |u_0|^{p-1} u_0 \phi_\epsilon v \to 0 \tag{75}$$

as $\epsilon \to 0$. We next show that

$$\int u_0^2 |\nabla u_0| |\nabla \phi_\epsilon| \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{76}$$

To see this, we remark that $\|\nabla\phi_\epsilon\|_{L^3(\mathbb{R}^3)} = \|\nabla\phi\|_{L^3(\mathbb{R}^3)}$. For any $\delta > 0$, we get

$$\begin{aligned}
\int u_0^2 |\nabla u_0| |\nabla \phi_\epsilon| &\leq \int_{|x|\leq\epsilon} \delta u_0^2 |\nabla\phi_\epsilon|^2 + \frac{4}{\delta} u_0^2 |\nabla u_0|^2 \\
&\leq \delta \|u_0\|_{L^6(\mathbb{R}^3)}^2 \|\nabla\phi_\epsilon\|_{L^3(\mathbb{R}^3)}^2 + \frac{4}{\delta}\int_{|x|\leq\epsilon} u_0^2 |\nabla u_0|^2 \\
&\to \delta \|u_0\|_{L^6(\mathbb{R}^3)}^2 \|\nabla\phi\|_{L^3(\mathbb{R}^3)}^2, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, we have (76). Similar arguments give (75). We have proved (74) for all $v \in C_0^\infty(\mathbb{R}^3)$. As in Section 4 we get a nontrivial nonnegative solution $u_0 \in H_r^1(\mathbb{R}^3)$ and thus the assertion.

We note that the above proof of (76) does not work in the case $N = 2$. Hence it is open whether an analogue to Theorem 7 holds for $N = 2$.

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