Nonlinear Schrödinger equations with unbounded and decaying radial potentials

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Abstract. We establish some embedding results of weighted Sobolev spaces of radially symmetric functions. The results then are used to obtain ground state solutions of nonlinear Schrödinger equations with unbounded and decaying radial potentials. Our work unifies and generalizes many existing partial results in the literature.

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1 Introduction

This paper is concerned with nonlinear Schrödinger type equations with potentials which may be unbounded, decaying and vanishing. These type equations have been studied recently (e.g., [1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 18, 19, 20, 24, 25, 26])

$$-\Delta u + V(x)u = f(x, u) \tag{1.1}$$

where the dependence on x can tend to infinity or zero somewhere. In contrast to the case of bounded spatial dependence on potentials which has been studied extensively (e.g., [7, 12, 14, 15, 22, 27] and references therein) and which requires basically a standard subcritical nonlinear growth condition

$$|f(x, u)| \leq C(1 + |u|^{p-1}), 2 < p < 2^* = \frac{2N}{N-2},$$

the case of potentials with growth and decay is more subtle in determining the optimal range of nonlinearity.

In this paper we shall focus on the following model equation

$$
(1.2) \qquad \begin{cases} -\Delta u + V(|x|)u = Q(|x|)u^{p-1}, \quad u > 0, \quad \text{in} \ \ \mathbb{R}^N \\[1ex] |u(x)| \to 0, \quad \text{as} \ \ |x| \to \infty. \end{cases}
$$

We establish the necessary functional framework in which solutions are naturally studied by variational methods and the existence and qualitative property of solutions can be examined. It turns out that it is the interplay between the growth of the nonlinearity and the rates of growth and decay for the potentials that determines the existence and non-existence of solutions. This will be done by studying the embedding between some weighted Sobolev spaces. We focus on the radially symmetric case for which stronger results can be established than for the general cases. To the best of our knowledge this type embedding results have not been studied in details (see Remarks 3 and 4 on some related results on existence of solutions by various methods). We refer to [17, 21] for general theory of weighted Sobolev spaces.

We always assume $N \geq 2$. We make the following assumptions.

(V) $V(r) \in C((0,\infty))$, $V(r)r^{N-1} \in L^1(0,1)$, $V(r) \geq 0$, and there exists $a > -2(N-1)$ such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C((0,\infty))$, $Q(r) > 0$, and there exist $b_0 > -2$ and $b \in \mathbb{R}$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty, \quad \limsup_{r\to\infty} \frac{Q(r)}{r^b} < \infty.$$

(VQ) There exist $-2 \geq a_0 > -N$, $b_0 > a_0$ such that

$$\liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0, \quad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

We introduce some notations. Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial}\}.$$

Let $D_r^{1,2}(\mathbb{R}^N)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Denote

$$H_r^1(\mathbb{R}^N; V) = \overline{C_{0,r}^\infty(\mathbb{R}^N)}^{\|\cdot\|_V},$$

where

$$\|u\|_V = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(|x|)u^2) dx\right)^{\frac{1}{2}}.$$

Define

$$L^p(\mathbb{R}^N; Q) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \mid \ u \,\text{is measurable}, \int_{\mathbb{R}^N} Q(|x|)|u|^p dx < \infty \right\}.$$

Similarly we may define $L^2(\mathbb{R}^N; V)$. Then $H_r^1(\mathbb{R}^N; V) = D_r^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$.

If (V) and (Q) are satisfied, we define

$$\overline{p} := \overline{p}(b_0) = \frac{2(N + b_0)}{N-2}, \quad b_0 > -2$$

$$\underline{p} := \underline{p}(a,b) = \begin{cases} \dfrac{2(2N-2+2b-a)}{2N-2+a}, & b \geq a > -2 \\[2ex] \dfrac{2(N+b)}{N-2}, & b \geq -2, -2 \geq a > -2(N-1) \\[2ex] 2, & a > -2(N-1), b \leq \max\{a, -2\}. \end{cases}$$

If in addition, (VQ) is satisfied, we define

$$\overline{p} := \overline{p}(a_0, b_0) = \frac{2(2N-2+2b_0-a_0)}{2N-2+a_0}, \quad -2 \geq a_0 > -N, \ b_0 > a_0.$$

Theorem 1. *Let $N \geq 3$. Assume* (V) *and* (Q)*. Then*

$$H_r^1(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q) \quad \text{for} \ \ \overline{p}(b_0) \geq p \geq \underline{p}(a,b). \tag{1.3}$$

Furthermore, if $b \geq \max\{a, -2\}$, the embedding is compact for $\overline{p} > p > \underline{p}$, and if $b < \max\{a, -2\}$, the embedding is compact for $\overline{p} > p \geq 2$. If in addition, (VQ) *is satisfied, $\overline{p}(b_0)$ can be replaced by $\max\{\overline{p}(b_0), \overline{p}(a_0, b_0)\}$, and the conclusions still hold.*

Theorem 2. *Let $N = 2$. Assume* (V) *and* (Q). *Then*

$$H^1_r(\mathbb{R}^2; V) \hookrightarrow L^p(\mathbb{R}^2; Q), \text{ for } \infty > p \geq \underline{p}. \tag{1.4}$$

Furthermore, if $b \geq a$, the embedding is compact for $\infty > p > \underline{p}$, and if $b < a$, the embedding is compact for $\infty > p \geq 2$.

Theorem 3. *Let $N \geq 2$. Assume* (V) *and* (Q) *or assume* (V), (Q) *and* (VQ) *with the corresponding $\overline{p}$ and $\underline{p}$ defined such that $\overline{p} > p > \underline{p}$. Then equation* (1.2) *has a ground state solution $u \in H^1_r(\mathbb{R}^N; V)$, namely*

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 + V(|x|)u^2}{\left(\displaystyle\int_{\mathbb{R}^N} Q(|x|)u^p\right)^{\frac{2}{p}}} = \inf_{\substack{v \in H^1_r(\mathbb{R}^N;V)\\ v \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2 + V(|x|)v^2}{\left(\displaystyle\int_{\mathbb{R}^N} Q(|x|)|v|^p\right)^{\frac{2}{p}}}.$$

Moreover, when $a > -2$, there exist $C, c > 0$ such that

$$u(x) \leq C \exp\left(-c|x|^{\frac{2+a}{2}}\right).$$

For $a \leq -2$, there exists $C > 0$ such that $u(x) \leq C|x|^{-\frac{N-2}{2}}$.

Remark 1. We show under (V),

$$\|u\|^2_V = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(|x|)u^2)dx$$

is a norm. Checking the proofs, the above three theorems remain valid as long as

$$\int_{B_R} |\nabla u|^2 + V(|x|)u^2 \geq \lambda_R \int_{B_R} u^2$$

for some $\lambda_R > 0$ for $R >> 1$, even $V(r)$ is negative somewhere. Also for the existence of solutions of (1.2), Q can be allowed to be zero somewhere (in this case $\displaystyle\int Q(|x|)|u|^p$ is not necessarily a norm anymore). Theorem 3 still holds as long as $Q(|x|) \not\equiv 0$.

Remark 2. For $a > -2$, we prove exponential decay of the solutions and thus these solutions are in $H^1(\mathbb{R}^N)$. For $-2(N-1) < a \leq -2$, we do not know whether the solutions have exponential decay.

Remark 3. Problems like (1.2) have been studied recently in [1, 25] without assuming radial symmetry. Comparing with these, our results give existence of ground state solutions for a wider range of nonlinearity (in terms of the range of p). For example, in [1] the authors consider $-\Delta u + V(x)u = Q(x)u^{p-1}$ and give the existence of a ground state in the weighted Sobolev space assuming that $\frac{A_1}{1+|x|^\alpha} \le V(x) \le A_2$, $0 < Q(x) \le \frac{k}{1+|x|^\beta}$ for positive constants A_1, A_2, k and $\alpha \in (0,2), \beta \ge 0$, and that $p^\# < p < 2^*$, where $p^\# := 2^* - \frac{4\beta}{\alpha(N-2)}$ for $0 < \beta < \alpha$, and $p^\# = 2$ otherwise. In our notation $a := -\alpha \in (-2,0)$, $b_0 = 0$, $b = -\beta \le 0$ and $p^\# = 2^* - \frac{4b}{a(N-2)}$, if $a < b < 0$ and $p^\# = 2$, otherwise. Our result gives a radial ground state for $\underline{p} = \frac{2(2N-2+2b-a)}{2N-2+a} < p < 2^*$ if $b > a$, and for $2 < p < 2^*$ otherwise. Note it always holds $\underline{p} < p^\#$. It is proved in [1] that in general for $p < p^\#$, a ground state does not exist. Our result gives a ground state in the radially symmetric class. Similar comparison can be done with the results in [25] where $a \ge 0$, $b \ge 0$ was considered without assuming radial symmetry and the range for the nonlinearity in terms of p is more restricted.

Remark 4. For equations like (1.2) with radial potentials, our results contain many existing results established by a variety of methods. Our results not only unify and generalize the existing results but also establish a unified framework. We follow the approach used by Sintzoff and Willem in [24] which considers ground states for homogeneous potentials with $a \ge 0$, $b \ge 0$. The result [12] by Ding-Ni is regarded as the case $a = a_0 = 0, b_0 = 0, b > 0$ and [12] used an approximation methods from bounded domains. We also mention the classical work [7, 27] on existence of ground states in radial classes for autonomous case or bounded spatial dependence. Related work can be found in [14, 15, 18, 19]. Souplet and Zhang in [26] give the existence of a radial solution for $a \in (-2,0)$ and $Q(|x|) \equiv 1$ by using the parabolic flow method without giving the least energy property. Our results extend and generalize these work on existence theory and also provide a unified functional framework for studying more general nonlinearity. Our approach is uniform in dealing with all the parameters of growth and decay, avoiding various ad hoc devices used in above mentioned papers for establishing existence of solutions.

2 The proofs of main Theorems

We start with stating a few known results and giving a few preliminary lemmas which we need. We use C_i to denote various constants independent of the functions, and for any set $A \subset \mathbb{R}^N$, A^c denotes the complement of A.

Lemma 1. a.) *Assume* (V). *Then there exists* $C > 0$ *such that for all* $u \in C_{0,r}^{\infty}(\mathbb{R}^N)$,

$$(2.1) \qquad |u(x)| \leq C||u||_{H_r^1(\mathbb{R}^N;V)}|x|^{-\frac{2(N-1)+a}{4}}, \quad |x| >> 1.$$

b.) Assume (VQ). *Then there exists* $r_0 > 0$ *and* $C > 0$ *such that for all* $u \in C_{0,r}^{\infty}(B_{r_0}(0))$,

$$(2.2) \qquad |u(x)| \leq C||u||_{H_r^1(\mathbb{R}^N;V)}|x|^{-\frac{2(N-1)+a_0}{4}}, \quad 0 < |x| < r_0.$$

(2.1) and (2.2) are improvements of Strauss Radial Lemma ([27]) and were proved for a homogeneous potential V with $a \geq 0$ in [24]. Using (V) and (VQ) the proof here is similar and we omit it.

Lemma 2. *Let* $N \geq 3$, $2 \leq p < \infty$, $p = \dfrac{2(N+c)}{N-2}$ *for some* $-2 \leq c < \infty$. *Then there exists* $C > 0$ *such that for all* $u \in \mathcal{D}_r^{1,2}(\mathbb{R}^N)$

$$(2.3) \qquad \left(\int_{\mathbb{R}^N} |x|^c |u|^p dx\right)^{\frac{2}{p}} \leq C \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

This result was due to [13], and was reproved by somewhat different arguments in [16, 23].

Lemma 3. *Let* $N \geq 2$, $1 \leq p \leq \infty$. *Then for any* $\infty > R > r > 0$ *with* $R >> 1$, *the following embedding is compact*

$$(2.4) \qquad H_r^1(B_R \backslash B_r; V) \hookrightarrow L^p(B_R \backslash B_r; Q).$$

Proof. Note first for $R > r > 0$ with $R >> 1$, the norm of $H^1_r(B_R \backslash B_r; V)$ is equivalent to the norm of $H^1_r(B_R \backslash B_r) = \{u \in H^1(B_R \backslash B_r) \mid u \text{ is radial}\}$. By Ascoli Theorem, $H^1_r(B_R \backslash B_r)$ is compactly embedded into $L^p(B_R \backslash B_r)$ for all $1 \leq p \leq \infty$. Finally, we note that $L^p(B_R \backslash B_r)$ is embedded into $L^p(B_R \backslash B_r; Q)$. □

Lemma 4. *The functional* $||u||_V = \sqrt{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(|x|)u^2)dx}$ *defines a norm so* $H^1_r(\mathbb{R}^N; V)$ *is well-defined. Moreover, for* $R >> 1$,

$$H^1_r(B_R; V) \hookrightarrow H^1(B_R), \tag{2.5}$$

where $H^1_r(B_R; V) = \left\{ u\big|_{B_R} \mid u \in H^1_r(\mathbb{R}^N; V) \right\}$.

Proof. If $||u||_V = 0$, then $\displaystyle\int |\nabla u|^2 = 0$ and u is a constant. It follows from (V), $\liminf_{r\to\infty} V(r) r^{-a} > 0$, one has $u = 0$.

Next, let $R_1 > 0$ be such that $(\text{supp} V)^c \subset B_{R_1}$. For $R \geq R_1 + 1$, we claim $H^1_r(B_R; V) \hookrightarrow H^1(B_R)$. For every $u \in H^1_r(B_R, V)$, we have $\displaystyle\int_{B_R \backslash B_{R_1}} u^2 \leq C_1 \int_{B_R} V u^2$. Choose a cut-off function $\varphi \in \mathcal{C}^\infty_0(B_R)$ satisfying $\varphi(x) \equiv 1$ for $|x| \leq R_1$. Then, by Poincaré inequality

$$\begin{aligned}
\int_{B_{R_1}} u^2 \leq \int_{B_R} (\varphi u)^2 &\leq C_2 \int_{B_R} |\nabla(\varphi u)|^2 \\
&\leq C_3 \left(\int_{B_R} |\nabla u|^2 + \int_{B_R \backslash B_{R_1}} u^2 \right) \\
&\leq C_4 \int_{B_R} |\nabla u|^2 + V u^2.
\end{aligned}$$

□

We need the Hardy inequality.

Lemma 5. *Let* $N \geq 3$. *For all* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}. \tag{2.6}$$

Lemma 6. ([28]) *Let $N = 2$. For all $u \in \mathcal{D}_0^{1,2}(B_1)$*

$$\int_{B_1} |\nabla u|^2 \geq \frac{1}{4} \int_{B_1} |x|^{-2} \left(\ln \frac{1}{|x|} \right)^{-2} u^2.$$

Proof of Theorem 1. We distinguish several cases.

Case 1. First we assume (V) and (Q) with $a \geq -2$. For the embedding, it suffices to show

$$S_r(V,Q) := \inf_{u \in H_r^1(\mathbb{R}^N;V)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 + V u^2}{\left(\displaystyle\int_{\mathbb{R}^N} Q|u|^p \right)^{\frac{2}{p}}} > 0.$$

If this is false, there exist $(u_n) \subset H_r^1(\mathbb{R}^N;V)$ such that $\displaystyle\int Q|u_n|^p \equiv 1$, $\displaystyle\int (|\nabla u_n|^2 + V u_n^2) = o(1)$. Writing $p = \dfrac{2N}{N-2} + \dfrac{2c}{N-2}$, by $p \leq \overline{p}$, we have $c \leq b_0$. For any $r > 0$ small enough, $Q(x) \leq C_0 |x|^{b_0}$, $|x| \leq r$, for some $C_0 > 0$. By Lemma 2,

$$\begin{aligned} \int_{B_r} Q|u_n|^p \quad & \leq C_0 \int_{B_r} |x|^{b_0} |u_n|^p \leq C_0 r^{b_0 - c} \int_{B_r} |x|^c |u_n|^p \\ & \leq C_0 r^{b_0-c} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{\frac{p}{2}} = o(1) \cdot r^{b_0 - c}. \end{aligned} \tag{2.7}$$

On the other hand, there exists $R_0 > 0$, for some $C_1, C_2 > 0$,

$$\begin{aligned} Q(x) \leq C_1 |x|^b \qquad & \text{for } |x| \geq R_0 \\ V(x) \geq C_2 |x|^a \qquad & \text{for } |x| \geq R_0. \end{aligned}$$

Then by Lemma 1 a., for $R > R_0$

$$\begin{aligned} \int_{B_R^c} Q|u_n|^p \quad & \leq \quad C_1 \int_{B_R^c} |x|^b |u_n|^p \\ & = \quad C_1 \int_{B_R^c} |x|^{b-a} |u_n|^{p-2} |x|^a |u_n|^2 \\ & \leq \quad C_1 R^{b-a-(p-2)(\frac{N-1}{2}+\frac{a}{4})} \cdot C_2^{-1} \int_{B_R^c} V(x) u_n^2 \\ & = \quad C_3 R^{b-a-(p-2)(\frac{N-1}{2}+\frac{a}{4})} \cdot o(1). \end{aligned} \tag{2.8}$$

Together with Lemma 3, we get $\int_{\mathbb{R}^N} Q|u_n|^p \to 0$, a contradiction.

Next, we consider compactness. Assume first $\overline{p} > p > \underline{p}$. Let $(u_n) \subset H_r^1(\mathbb{R}^N; V)$ be such that $||u_n|| \leq C$. Without loss of generality, we may assume $u_n \rightharpoonup 0$. We claim $u_n \to 0$ in $L^p(\mathbb{R}^N; Q)$. As in (2.7), we get

$$\int_{B_r} Q|u_n|^p \leq C_0 r^{b_0 - c} ||u_n||^p_{H_r^1(\mathbb{R}^N;V)} \tag{2.9}$$

and as in (2.8), we get

$$\int_{B_R^c} Q|u_n|^p \leq C_3 R^{b-a-(p-2)(\frac{N-1}{2}+\frac{a}{4})} ||u_n||^2_{H_r^1(\mathbb{R}^N;V)}. \tag{2.10}$$

Since $b_0 - c > 0$, $b - a - (p-2)\left(\dfrac{N-1}{2} + \dfrac{a}{4}\right) < 0$, together with Lemma 3, we get $\int_{\mathbb{R}^N} Q|u_n|^p \to 0$, as $n \to \infty$.

Finally, we consider the compactness for the case $b < a$ and $p = 2$. Similar to (2.10) we have

$$\int_{B_R^c} Q|u_n|^2 \leq C_3 R^{b-a} ||u_n||^2_{H_r^1(\mathbb{R}^N;V)}. \tag{2.11}$$

By Lemma 5,

$$\int_{B_r} Q|u_n|^2 \leq C_0 \int_{B_r} |x|^{-2} u_n^2 |x|^{2+b_0} \leq C_1 r^{2+b_0} \int_{\mathbb{R}^N} |\nabla u_n|^2. \tag{2.12}$$

Since $b_0 > -2$ and $b < a$ we get $\int_{\mathbb{R}^N} Q|u_n|^p \to 0$, as $n \to \infty$. Case 1 is finished.

Case 2. We assume (V) and (Q) with $-2(N-1) < a \leq -2$. Checking the proofs above when $b \geq -2$, we see (2.7), (2.9) still hold and we need to get estimates like (2.8) and (2.10). Writing $p = \frac{2N+2c}{N-2}$ we get $c \geq b$. Then for $R > R_0$, we use Lemma 2 to get

$$\begin{aligned} \int_{B_R^c} Q|u_n|^p &\leq C_1 \int_{B_R^c} |x|^b |u_n|^p \\ &= C_1 \int_{B_R^c} |x|^{b-c} |x|^c |u_n|^p \\ &\leq C_3 R^{b-c} \cdot ||\nabla u_n||_2^p. \end{aligned} \tag{2.13}$$

Using $b - c \leq 0$ we get the estimate as for (2.8) and (2.10).

When $b < -2$ we need to prove compactness for $p = 2$. Let ϕ be a cut-off function such that $\phi(r) = 1$ for $r \geq 2R_0$ and $\phi(r) = 0$ for $r \leq R_0$. Then by Lemma 5,

$$\int_{B_R^c} Q|u_n|^2 \leq C_1 R^{b+2} ||\nabla(\phi u_n)||_2^2 \leq C_2 R^{b+2} ||\nabla u_n||_2^2.$$

Since (2.12) still holds and $b < -2$, the remaining part of the proof is the same as before. Case 2 is done.

Case 3. We assume (V), (Q), and (VQ) are satisfied. Checking the proofs above we need to get estimates similar to (2.7)–(2.12). We observe that depending on a, (2.8), (2.10) or (2.13) still hold. We need to get estimates near 0. By (VQ) there is $r_0 > 0$ such that for some $C_0 > 0$, $V(r) \geq C_0 r^{a_0}$ for $0 < r < r_0$. We choose a cut-off function ϕ such that $\phi(r) = 1$ for $0 \leq r \leq \frac{r_0}{2}$, and $\phi(r) = 0$ for $r \geq r_0$. Then by Lemma 1 b., for $r < r_0/2$

$$\begin{aligned} \int_{B_r} Q|u_n|^p \quad & \leq C_0 \int_{B_r} |x|^{b_0} |\phi u_n|^p \\ & \leq C_0 r^{b_0 - a_0 - (\frac{N-1}{2} + \frac{a_0}{4})(p-2)} ||\phi u_n||^{p-2}_{H_r^1(\mathbb{R}^N;V)} \int_{B_{r_0}} |x|^{a_0} |u_n|^2 \\ & \leq C_1 r^{b_0 - a_0 - (\frac{N-1}{2} + \frac{a_0}{4})(p-2)} ||u_n||^p_{H_r^1(\mathbb{R}^N;V)}. \end{aligned}$$

Since $b_0 - a_0 - (\frac{N-1}{2} + \frac{a_0}{4})(p-2) > 0$ we get the desired estimate like (2.7) and (2.9). For (2.12) we note

$$\int_{B_r} Q|u_n|^2 \leq C_0 \int_{B_r} |x|^{a_0} u_n^2 |x|^{b_0 - a_0} \leq C_1 r^{b_0 - a_0} ||u_n||^2_{H_r^1(\mathbb{R}^N;V)}.$$

Since $b_0 > a_0$, the remaining part of the proof is the same as before. Case 3 is proved. □

Proof of Theorem 2. Checking the proof of Theorem 1, the term on B_R^c can be treated similarly as in (2.8) and (2.10). If $b_0 > 0$, $p \geq 2$, by Lemma 4,

$$\begin{aligned} \int_{B_r} Q(|x|)|u_n|^p \quad & \leq \quad C_0 r^{b_0} \int_{B_r} |u_n|^p \leq C_0 r^{b_0} \int_{B_1} |u_n|^p \\ & \leq \quad C_1 r^{b_0} ||u_n||^{\frac{p}{2}}_{H_r^1(\mathbb{R}^2;V)}. \end{aligned}$$

If $b_0 \in (-2, 0]$, $p \geq 2$, we choose $\delta > 0$ such that $b_0 - \delta > -2$. Choose a cut-off function $\varphi \in C_0^\infty(B_1)$, such that $\varphi(x) \equiv 1$ for $|x| \le \frac{1}{2}$. Then by Lemma 6

$$\begin{aligned}
&\int_{B_r} Q|u_n|^p \\
\le\ & C_0 \int_{B_r} |x|^{b_0-\delta} \left(\ln \frac{1}{|x|}\right)^{b_0-\delta} (u_n\varphi)^{-b_0+\delta} |x|^\delta \left(\ln \frac{1}{|x|}\right)^{\delta-b_0} |u_n|^{p+b_0-\delta} \\
\le\ & C_1 r^\delta \left(\ln \frac{1}{r}\right)^{\delta-b_0} \left(\int_{B_1} |x|^{-2} \left(\ln \frac{1}{|x|}\right)^{-2} (u_n\varphi)^2\right)^{\frac{-b_0+\delta}{2}} \left(\int_{B_r} |u_n|^{\frac{2(p+b_0-\delta)}{2+b_0-\delta}}\right)^{\frac{2+b_0-\delta}{2}} \\
\le\ & C_2 r^\delta \left(\ln \frac{1}{r}\right)^{\delta-b_0} ||u_n||^{\frac{p}{2}}_{H^1_r(\mathbb{R}^2;V)}.
\end{aligned}$$

Finally, for $b < a$, we consider $p = 2$. Again, if $b_0 > 0$, by Lemma 4,

$$\int_{B_r} Q|u_n|^2 \le C_0 r^b \int_{B_r} u^2 \le C_0 r^{b_0} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V u_n^2.$$

If $b_0 \in (-2, 0]$, we choose $\delta > 0$, $b_0 - \delta > -2$. Then by Lemma 6,

$$\begin{aligned}
\int_{B_r} Q|u_n|^2 \ &\le\ C_0 r^\delta \left(\ln \frac{1}{\delta}\right)^2 \int_{B_r} |x|^{-2} \left(\ln \frac{1}{|x|}\right)^{-2} u_n^2 \\
&\le\ C_1 r^\delta \left(\ln \frac{1}{\delta}\right)^2 \int_{\mathbb{R}^2} |\nabla u_n|^2 + V u_n^2.
\end{aligned}$$

□

Proof of Theorem 3. The existence of a ground state solution follows from the compact embedding immediately.

Next we show the decay property in case $a > -2$. Let $C_1 > 0$, $R_1 > 0$ be such that

$$V(|x|) \ge C_1 |x|^a, \text{ for } |x| \ge R_1.$$

Consider $\varphi(r) = \exp(-c r^{\frac{2+a}{2}})$ with $c = \dfrac{\sqrt{2C_1}}{2+a} > 0$. Then a direct computation shows there is $R_2 > 0$, for $|x| > R_2$,

$$-\Delta\varphi + V(|x|)\varphi \ge \frac{C_1}{2} |x|^a \varphi.$$

By Lemma 1 a., there is $R_3 > 0$, for $|x| \ge R_3$,

$$Q(x)|u(x)|^{p-2} \le \frac{C_1}{2} |x|^a.$$

Then we get $-\Delta(u-\varphi)+(V(|x|)-\frac{C_1}{2}|x|^a)(u-\varphi)\le 0$, for $|x|\ge R_3$. From this we get

$$u(x)\le\varphi(x) \text{ for } |x|\ge R_3.$$

Finally, when $-2(N-1)<a\le -2$, the decay property follows directly from Lemma 1 b.. □

3 Further results and remarks

We finish the paper with some discussions on further results and relations with other work.

3.1 With the embedding theorems established in this paper, we can study the existence of solutions for more general equations like (1.1). This would follow from some rather standard techniques. For simplicity, let us state a result for the following equation with the proof omitted

$$(3.1)\qquad \begin{cases} -\Delta u+V(|x|)u=Q(|x|)f(u), \text{ in } \mathbb{R}^N \\ |u(x)|\to 0, \text{ as } |x|\to\infty. \end{cases}$$

We assume (V) and (Q), or assume (V), (Q) and (VQ) with the corresponding $\underline{p}$ and $\overline{p}$ are defined as in Section 1 such that $\underline{p}<\overline{p}$. We assume $f\in C(\mathbb{R},\mathbb{R})$, $f(0)=0$; there exists $C>0$, $\underline{p}<p_1\le p_2<\overline{p}$ such that

$$|f(u)|\le C(|u|^{p_1-1}+|u|^{p_2-1});$$

and there exists $\mu>2$ such that

$$0<\mu F(u)\le uf(u), \qquad \forall u\in\mathbb{R}.$$

Here $F(u)=\displaystyle\int_0^u f(s)ds$. Then we have

Theorem 4. *Under the above conditions, equation* (3.1) *has a positive solution. If in addition, f is odd in u,* (3.1) *has infinitely many solutions. All these solutions are in $H_r^1(\mathbb{R}^N;V)$, and satisfy the decay property in Theorem 3.*

Again, the result still holds for V slightly negative and Q vanishing somewhere.

3.2 When $Q(r)$ is exponentially flat near zero, b_0 can be taken arbitrarily large. Thus we obtain solution for (1.2) for all $p > \underline{p}$. For a problem in a bounded domain this was observed in [18]. In a recent paper [2], the authors consider $-\epsilon^2 \Delta u + V(|x|)u = u^p, x \in \mathbb{R}^N$ and under the condition that a weighted potential has local maximum or local minimum they constructed radially symmetric solutions for all $p > 1$ as long as ϵ is sufficiently small. Note that the solution given in [2] in general is not ground state solution in the radially symmetric class. A related result to [2] is given in [10] which allows the potentials to be zeroes somewhere.

3.3 Our results can be used to deduce existence of nonradial positive solutions for

$$\begin{cases} -\Delta u + V(|x|)u = Q(|x|)u^{p-1}, u > 0, \text{ in } B_R \\ \\ u = 0, \text{ on } \partial B_R \end{cases} \tag{3.2}$$

for R sufficiently large. This follows from the idea used in [12] (see also [15, 24]) and it is rather standard by now. Let us sketch the process a bit here.

Define $H^1(\mathbb{R}^N; V)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$||u||_V^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + V(|x|)u^2,$$

and define

$$\begin{aligned} S(V,Q) &= \inf \left\{ \int |\nabla u|^2 + V(|x|)u^2 \mid u \in H^1(\mathbb{R}^N; V), \int Q(|x|)|u|^p = 1 \right\} \\ S_r(V,Q) &= \inf \left\{ \int |\nabla u|^2 + V(|x|)u^2 \mid u \in H_r^1(\mathbb{R}^N; V), \int Q(|x|)|u|^p = 1 \right\}. \end{aligned}$$

Then $S(V,Q) \leq S_r(V,Q)$. We can also define $S(B_R; V, Q)$ and $S_r(B_R; V, Q)$ using $H_0^1(B_R; V)$ and $H_{0,r}^1(B_R; V)$. It is easy to show

$$\begin{aligned} &\lim_{R \to \infty} S(B_R; V, Q) = S(V,Q) \\ &\lim_{R \to \infty} S_r(B_R; V, Q) = S_r(V,Q). \end{aligned}$$

If for some p, $S(V,Q) < S_r(V,Q)$, we then get that for large R, $S(B_R; V, Q) < S_r(B_R; V, Q)$. Then it follows that (3.2) has both radial and nonradial solutions. For example, following the arguments as in [1], we can show that when $0 > b > a$, $S(V,Q) = 0$ for $p < p^\# := \dfrac{2N}{N-2} - \dfrac{4b}{a(N-2)}$ and for $p > \dfrac{2N}{N-2}$. On the other hand, our result shows $S_r(V,Q) > 0$ for $\underline{p} \le p \le \overline{p}$. Note $\underline{p} < p^{\#}$, $\overline{p} > \dfrac{2N}{N-2}$ for $b_0 > 0$. Then we obtain that when $\underline{p} < p < p^{\#}$ or $\frac{2N}{N-2} < p < \overline{p}$, for $R >> 1$, (3.2) has both radial and nonradial solutions. Similar results can be stated for general cases, we leave it to interested readers.

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